Lecture 2: Mixed Strategy and Nash Equilibrium

Algorithmic and Mathematical Foundations of Game Theory and Economics
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1 Pure and Mixed Strategies

Recall that a normal-form game is specified by (1) each player's action space, and (2) the payoffs to the players at each joint action profile. In this lecture, we will start to introduce the strategy of a player, which specifies how the player chooses her action.

Let's use the Rock-Paper-Scissors game as our guiding example. If this game is played only a few times, *luck* will determine the outcome and no mathematical analysis seems plausible.[1] This suggests us to consider the setting of *repeated games*, where the same game is played for many times, and the mathematical analysis focuses on the average payoffs of the players.

In the repeated games setting, a player's strategy may stick to one action. This is called *pure strategy*. Pure strategy is used if one action is always better than the others. For instance, in the Prisoner Dilemma game of last lecture, "testify" is always the better option for Alice no matter what Bob chooses, so rationally she should adopt the pure strategy of "testify".

However, in most other games, including even the simple Rock-Paper-Scissors game, it is clearly unwise for a player to limit herself solely to pure strategies. As conventional wisdom suggests, she should introduce *randomness* into her choices of actions. This motivates the concept of *mixed strategy*.

Definition 1.1. Let $\mathcal{A}_i$ denote the action space of player i, where $|\mathcal{A}_i|$ is finite. A mixed strategy of player i is a probability distribution over $\mathcal{A}_i$. A pure strategy is a special case of mixed strategy, where one of the actions in $\mathcal{A}_i$ is chosen with probability 1.

Following standard notation in probability theory, we use $\Delta(\mathcal{A}_i)$ to denote the set of mixed strategies of player i:

$$\Delta(\mathcal{A}_i) \;:=\; \left\{ \mathbf{x}_i \in \mathbb{R}^{|\mathcal{A}_i|} \;\middle|\; \sum_{j \in \mathcal{A}_i} x_{ij} = 1 \;\text{ and }\; \forall j \in \mathcal{A}_i,\; x_{ij} \geq 0 \right\}.$$

We use $\mathbf{x}_i \in \Delta(\mathcal{A}_i)$ to denote a mixed strategy of player i. A mixed strategy is a column vector, but writing a column vector within text is space-wasting. Thus, we abuse notation slightly by writing a mixed strategy within text as a row vector, say $\mathbf{x}_i = (\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$, but we *actually* mean the column vector $\mathbf{x}_i = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{6} \end{bmatrix}$. Also, we use

$$\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) \;\in\; \Delta(\mathcal{A}_1) \times \Delta(\mathcal{A}_2) \times \ldots \times \Delta(\mathcal{A}_n)$$

to denote a joint mixed strategy profile in an n-player game.

When we want to focus from the perspective of a particular player i, we will write $\mathbf{x}$ as $(\mathbf{x}_i, \mathbf{x}_{-i})$, where $\mathbf{x}_i$ is the mixed strategy of player i, and $\mathbf{x}_{-i}$ collects the mixed strategies of the other players. If player i changes her mixed strategy from $\mathbf{x}_i$ to $\mathbf{x}'_i$ while the other players remain the same, the new joint mixed strategy profile is written as $(\mathbf{x}'_i, \mathbf{x}_{-i})$.

[1] However, there may be other analyses like psychological or behavioral ones. For instance, some children always pick Paper in their first game.

Recall that in a normal-form game, the payoff function of player i is denoted by $u_i : \mathcal{A} \to \mathbb{R}$. Be reminded that the domain of u_i is $\mathcal{A}$, the space of joint action profiles. When the players adopt mixed strategies in a joint mixed strategy profile $\mathbf{x}$, the payoff function u_i is extended naturally via expectation, where the underlying probability distribution is that each player ℓ chooses *independently* her action a_ℓ following her mixed strategy $\mathbf{x}_\ell$:

$$u_i(\mathbf{x}) \;:=\; \mathbb{E}_{(a_1,a_2,\ldots,a_n)\sim(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n)}\left[u_i(a_1,a_2,\ldots,a_n)\right] \;.$$

Example 1.2. In the Rock-Paper-Scissors game, suppose Alice (player 1) uses mixed strategy $\mathbf{x}_1 = (\frac{1}{3}, \frac{2}{3}, 0)$, i.e., she chooses Rock with probability $\frac{1}{3}$, Paper with probability $\frac{2}{3}$, and Scissors with probability 0. Suppose Bob (player 2) uses the mixed strategy $\mathbf{x}_2 = (\frac{8}{15}, \frac{2}{15}, \frac{1}{3})$.

The joint mixed strategy profile $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$ corresponds to the probability distribution where Alice chooses her action randomly following $\mathbf{x}_1$, Bob chooses his action randomly following $\mathbf{x}_2$, and their random choices are independent. For instance, the joint action profile (R, P) is chosen with probability $\frac{1}{3} \times \frac{2}{15} = \frac{2}{45}$. The probabilities of all joint action profiles are given below:

		Bob's action		
		R	P	S
Alice's action	R	$\frac{8}{45}$	$\frac{2}{45}$	$\frac{1}{9}$
	P	$\frac{16}{45}$	$\frac{4}{45}$	$\frac{2}{9}$
	S	0	0	0

Recall that the payoff functions u_i are extended to mixed strategies via expectation. Using the above probabilities, we compute $u_1(\mathbf{x})$ as below:

$$\begin{aligned} u_1(\mathbf{x}) \;&=\; \frac{8}{45} \times 0 + \frac{2}{45} \times (-1) + \frac{1}{9} \times 1 \\ &\quad + \frac{16}{45} \times 1 + \frac{4}{45} \times 0 + \frac{2}{9} \times (-1) \;=\; \frac{1}{5}. \end{aligned}$$

Analogously, we can compute $u_2(\mathbf{x}) = -\frac{1}{5}$.

Recall that for two-player game, the payoffs to the two players can be specified by two matrices $\mathbf{A}, \mathbf{B}$. When the mixed strategies of the two players are $\mathbf{x}_1, \mathbf{x}_2$ respectively, it is not hard to see that the expected payoffs can be written compactly using linear algebra: $u_1(\mathbf{x}) = (\mathbf{x}_1)^\mathsf{T}\mathbf{A}\mathbf{x}_2$ and $u_2(\mathbf{x}) = (\mathbf{x}_1)^\mathsf{T}\mathbf{B}\mathbf{x}_2$, where $(\mathbf{x}_1)^\mathsf{T}$ is the transpose of the column vector $\mathbf{x}_1$. For instance, in the example above, $u_1(\mathbf{x})$ can be computed more conveniently as below:

$$u_1(\mathbf{x}) \;=\; \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{8}{15} \\ \frac{2}{15} \\ \frac{1}{3} \end{bmatrix} \;=\; \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \\ -\frac{2}{5} \end{bmatrix} \;=\; \frac{1}{5}\;.$$

2 Best Response

Many economics models rely on the assumption is rational self-interest: each individual aims to optimize her own benefit, without caring about the benefits of the others.[2] In other words, each individual is only interested in making best response to the other individuals' choices.

[2] One might argue that numerous real-world scenarios refute this assumption. This is certainly true. Nevertheless, the art of effective modeling lies in achieving a delicate balance between tractability and applicability. While acknowledging that the assumptions underpinning a simplified model may not perfectly reflect the nuances of a specific application, their sufficient proximity to reality permits rigorous mathematical analysis, and consequently, has potential for extracting valuable insights. This echoes the well-known adage of statistician George Box, who famously remarked, "All models are wrong, some are useful." This underscores that the inherent limitations of simplified models do not negate their potential to contribute to our understanding of complex phenomena.

Example 2.1. We revisit Example 1.2. We will determine whether the two players are making best responses.

Suppose Alice's mixed strategy is (a, b, c) where a, b, c are non-negative numbers which sum to 1, while Bob's mixed strategy is $\mathbf{x}_2 = (\frac{8}{15}, \frac{2}{15}, \frac{1}{3})$. The expected payoff of Alice is

$$\begin{bmatrix} a & b & c \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{8}{15} \\ \frac{2}{15} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} a & b & c \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{5} \\ \frac{1}{5} \\ -\frac{2}{5} \end{bmatrix} = \frac{a+b-2c}{5} .$$

To maximize $\frac{a+b-2c}{5}$, Alice can choose any $a, b \geq 0$ such that $a + b = 1$, and $c = 0$. $\mathbf{x}_1 = (\frac{1}{3}, \frac{2}{3}, 0)$ fulfills this, so $\mathbf{x}_1$ is a best response.

Next, suppose Bob's mixed strategy is (a, b, c) where a, b, c are non-negative numbers which sum to 1, while Alice's mixed strategy is $\mathbf{x}_1 = (\frac{1}{3}, \frac{2}{3}, 0)$. The expected payoff of Bob is

$$\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{b+c-2a}{3} .$$

To maximize $\frac{b+c-2a}{3}$, Bob can choose any $b, c \geq 0$ such that $b + c = 1$, and $a = 0$. In other words, player 2 can choose another mixed strategy, say $(0, \frac{1}{2}, \frac{1}{2})$, to get an expected payoff of $\frac{1}{3}$, which is higher than $-\frac{1}{5}$ he got previously. Thus, $\mathbf{x}_2$ is not a best response.

3 Nash Equilibrium — "Mutual Best Response"

Economists are interested in analyzing the outcomes of economic systems under the rational self-interest assumption. In some models, the focus lies on identifying *stable outcome* or *equilibrium*, where no individual has incentive to deviate from their current decision. In other words, *every* individual makes best response to the choices of the others. This underpins the definition of *Nash equilibrium*, formally stated below.

Definition 3.1. A joint mixed strategy profile $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ of a normal-form n-player game is a Nash equilibrium if for every player i, the mixed strategy $\mathbf{x}_i$ maximizes her own expected payoff, assuming all other players stick with their mixed strategies in $\mathbf{x}_{-i}$; precisely,

$$\mathbf{x}_i \in \arg\max_{\mathbf{x}'_i \in \Delta(\mathcal{A}_i)} u_i(\mathbf{x}'_i, \mathbf{x}_{-i}) .$$

If every strategy in a Nash equilibrium is a pure strategy, it is called a pure Nash equilibrium.

In Example 2.1, Alice makes best response but Bob does not, so $(\mathbf{x}_1, \mathbf{x}_2)$ in the example is not a Nash equilibrium. For a joint mixed strategy profile to be a Nash equilibrium, *every* player must make best response. This is a rather strong requirement, and it is unclear whether a Nash equilibrium exists or not. One of the most celebrating results in game theory is that Nash equilibrium exists under very general conditions. We will discuss the proof this result in the next lecture.

After defining Nash equilibrium, the two immediate questions are

1. How can we determine whether a joint mixed strategy profile is Nash equilibrium or not?
2. How can we compute a Nash equilibrium of any given game?

It turns out question 2 is not easy to answer. Indeed, we will devote a significant portion of this course to look into some algorithms for computing Nash equilibrium. Question 1 is much easier to answer. The following proposition provides a simple method to check whether a profile is a Nash equilibrium: at a Nash equilibrium, every player will choose random actions only among those that yield her the maximum expected payoff.

Proposition 3.2. Let $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ be a joint mixed strategy profile. For each action j of player i, let $v_{ij}(\mathbf{x}_{-i}) := u_i(\mathbf{e}_{ij}, \mathbf{x}_{-i})$, where $\mathbf{e}_{ij}$ is the pure strategy of player i choosing action j with probability 1. Let $\hat{v}_i(\mathbf{x}_{-i}) := \max_{j \in \mathcal{A}_i} v_{ij}(\mathbf{x}_{-i})$.

The profile $\mathbf{x}$ is a Nash equilibrium if and only if for every player i and every action $j \in \mathcal{A}_i$ where $x_{ij} > 0$, we have $v_{ij}(\mathbf{x}_{-i}) = \hat{v}_i(\mathbf{x}_{-i})$. In the rest of this course, this is called the best response condition (BR condition) of player i.

Proof: Since the underlying probability distribution is each player ℓ choosing her action randomly and independently following $\mathbf{x}_\ell$, we have

$$u_i(\mathbf{x}_i, \mathbf{x}_{-i}) \;=\; \sum_{j \in \mathcal{A}_i} x_{ij} \cdot u_i(\mathbf{e}_{ij}, \mathbf{x}_{-i}) \;=\; \sum_{j \in \mathcal{A}_i} x_{ij} \cdot v_{ij}(\mathbf{x}_{-i}) \;,$$

whereas $\sum_{j \in \mathcal{A}_i} x_{ij} = 1$. By the definition of $\hat{v}_i(\mathbf{x}_{-i})$, we have

$$\sum_{j \in \mathcal{A}_i} x_{ij} \cdot v_{ij}(\mathbf{x}_{-i}) \;\leq\; \sum_{j \in \mathcal{A}_i} x_{ij} \cdot \hat{v}_i(\mathbf{x}_{-i}) \;=\; \hat{v}_i(\mathbf{x}_{-i}) \;,$$

so the maximum possible value of $u_i(\mathbf{x}_i, \mathbf{x}_{-i})$, while keeping $\mathbf{x}_{-i}$ fixed, is $\hat{v}_i(\mathbf{x}_{-i})$. If $x_{ij} > 0$ implies $v_{ij}(\mathbf{x}_{-i}) = \hat{v}_i(\mathbf{x}_{-i})$, then

$$u_i(\mathbf{x}_i, \mathbf{x}_{-i}) \;=\; \sum_{j : x_{ij} > 0} x_{ij} \cdot \hat{v}_i(\mathbf{x}_{-i}) \;=\; \hat{v}_i(\mathbf{x}_{-i}) \;,$$

i.e., $\mathbf{x}_i$ is a mixed strategy of player i that achieves the maximum possible payoff.

Otherwise, there exists an action $k \in \mathcal{A}_i$ such that $x_{ik} > 0$ but $v_{ik}(\mathbf{x}_{-i}) < \hat{v}_i(\mathbf{x}_{-i})$. Then we have $x_{ik} \cdot v_{ik}(\mathbf{x}_{-i}) < x_{ik} \cdot \hat{v}_i(\mathbf{x}_{-i})$, and hence

$$\begin{aligned} u_i(\mathbf{x}_i, \mathbf{x}_{-i}) \;&=\; x_{ik} \cdot v_{ik}(\mathbf{x}_{-i}) + \sum_{j \in \mathcal{A}_i \setminus \{k\}} x_{ij} \cdot v_{ij}(\mathbf{x}_{-i}) \\ &<\; x_{ik} \cdot \hat{v}_i(\mathbf{x}_{-i}) + \sum_{j \in \mathcal{A}_i \setminus \{k\}} x_{ij} \cdot \hat{v}_i(\mathbf{x}_{-i}) \;=\; \hat{v}_i(\mathbf{x}_{-i}) \;, \end{aligned}$$

i.e., $\mathbf{x}_i$ falls short from achieving the maximum possible payoff. □

Example 3.3. Your instinct should have already told you that a Nash equilibrium of the two-player Rock-Paper-Scissors game is each player choosing the uniform distribution: $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2) = \left(\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right)$. Let's confirm this instinct formally. When $\mathbf{x}_2$ is fixed, let $\mathbf{e}_{1\mathrm{R}}$ denote the pure strategy of player i choosing Rock, then

$$u_1(\mathbf{e}_{1\mathrm{R}}, \mathbf{x}_2) \;=\; x_{2\mathrm{R}} \cdot 0 + x_{2\mathrm{P}} \cdot (-1) + x_{2\mathrm{S}} \cdot 1 \;=\; \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot (-1) + \frac{1}{3} \cdot 1 \;=\; 0.$$

Analogously, we compute $u(\mathbf{e}_{1\mathrm{P}}, \mathbf{x}_2) = u(\mathbf{e}_{1\mathrm{S}}, \mathbf{x}_2) = 0$. In the notations of Proposition 3.2, $\hat{v}_1(\mathbf{x}_{-1}) = 0$. It is easy to check that the BR condition of player 1 is satisfied: for each action j of player 1, $x_{1j} > 0$ implies $u(\mathbf{e}_{1j}, \mathbf{x}_2) = \hat{v}_1(\mathbf{x}_{-1}) = 0$. Due to the symmetry among the two players, the BR condition of player 2 is also satisfied. Thus, $\mathbf{x}$ is a Nash equilibrium.

Test Your Intuition: Suppose we twist the game as below: if one player chooses Paper and the other chooses Rock, the winner gets \$2 from her opponent, instead of \$1. Will the players choose Paper more often (i.e., with probability more than $\frac{1}{3}$) at the Nash equilibrium of the twisted game?

Example 3.4. Consider the two-player game represented by the following bimatrix:

$$\begin{bmatrix} (5,4) & (0,7) & (2,-1) \\ (4,8) & (1,6) & (6,10) \\ (3,6) & (3,3) & (4,3) \end{bmatrix}.$$

As before, we separate the bimatrix into $\mathbf{A} = \begin{bmatrix} 5 & 0 & 2 \\ 4 & 1 & 6 \\ 3 & 3 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 4 & 7 & -1 \\ 8 & 6 & 10 \\ 6 & 3 & 3 \end{bmatrix}$.

First, we check whether $\mathbf{x} = \left(\left(0, \frac{2}{3}, \frac{1}{3}\right), \left(0, \frac{1}{2}, \frac{1}{2}\right)\right)$ is a Nash equilibrium. To calculate $v_{1j}(\mathbf{x}_{-1})$ for all actions j, we compute the following matrix-vector product:

$$\mathbf{A}\mathbf{x}_2 = \begin{bmatrix} 5 & 0 & 2 \\ 4 & 1 & 6 \\ 3 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{7}{2} \\ \frac{7}{2} \end{bmatrix}.$$

Thus, $\hat{v}_1(\mathbf{x}_{-1}) = \frac{7}{2}$. Observe that $x_{12}, x_{13} > 0$, and the corresponding $v_{12}(\mathbf{x}_{-1}), v_{13}(\mathbf{x}_{-1})$ both equal to $\hat{v}_1(\mathbf{x}_{-1})$. The BR condition of player 1 is satisfied.

Next, to calculate $v_{2j}(\mathbf{x}_{-2})$ for all actions j, we compute the following matrix-vector product:

$$\mathbf{B}^\mathsf{T}\mathbf{x}_1 = \begin{bmatrix} 4 & 8 & 6 \\ 7 & 6 & 3 \\ -1 & 10 & 3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{22}{3} \\ 5 \\ \frac{23}{3} \end{bmatrix}.$$

Thus, $\hat{v}_2(\mathbf{x}_{-2}) = \frac{23}{3}$. The BR condition of player 2 is not satisfied, since $x_{22} = \frac{1}{2} > 0$ but $v_{22}(\mathbf{x}_{-2}) = 5 < \hat{v}_2(\mathbf{x}_{-2})$. Thus, $\mathbf{x}$ is not a Nash equilibrium.

We check whether $\mathbf{y} = \left(\left(\frac{1}{2}, 0, \frac{1}{2}\right), \left(\frac{3}{5}, \frac{2}{5}, 0\right)\right)$ is a Nash equilibrium. As before, we compute $\mathbf{A}\mathbf{y}_2 = \begin{bmatrix} 3 \\ \frac{14}{5} \\ 3 \end{bmatrix}$ and $\mathbf{B}^\mathsf{T}\mathbf{y}_1 = \begin{bmatrix} 5 \\ 5 \\ 1 \end{bmatrix}$. Observe the followings, which confirms $\mathbf{y}$ is a Nash equilibrium:

- $\hat{v}_1(\mathbf{y}_{-1}) = 3$, $y_{11}, y_{13} > 0$ and the corresponding $v_{11}(\mathbf{y}_{-1}), v_{13}(\mathbf{y}_{-1})$ both equal to $\hat{v}_1(\mathbf{y}_{-1})$, so the BR condition of player 1 is satisfied.
- $\hat{v}_2(\mathbf{y}_{-2}) = 5$, $y_{21}, y_{22} > 0$ and the corresponding $v_{21}(\mathbf{y}_{-2}), v_{22}(\mathbf{y}_{-2})$ both equal to $\hat{v}_2(\mathbf{y}_{-2})$, so the BR condition of player 2 is satisfied too.

Test Your Intuition: Can you find out another Nash equilibrium of the game? What are the differences of the payoffs between the two Nash equilibria?

As the "Test Your Intuition" of the previous example indicates, a game can be more than one Nash equilibria, and their payoffs can be quite different. Indeed, a n-player game can have $\Theta(2^n)$ Nash equilibria with vastly different payoffs, as the next example demonstrates.

Example 3.5. In an n-player game, each player can choose an action of either 0 or 1. All players receive the same payoff, determined as follows. Let s denote the sum of their chosen numbers. If s is odd, each player receives \$0. Otherwise, each player receives \$$s$.

We claim that any joint pure strategy profile at which there are even number of players choosing 1 is a pure Nash equilibrium. To see why, if a player sticks to her pure strategy, she receives \$$s$. If she switches, she receives \$0. As $s \geq 0$, her pure strategy is the unique best response of her.

By a simple counting, there are 2^{n-1} such joint pure strategy profiles. The *worst* pure Nash equilibrium is when all players choose 0, at which each player receives \$0. The *best* pure Nash equilibrium is the maximum possible even number of players choose 1, at which each player receives \$$n$ or \$$(n-1)$, depending on the parity of n.

4 Algorithms for Computing Nash Equilibrium of Some Two-Player Games

As we shall see later in this course, computing Nash equilibrium, even restricted to two-player games, is not simple. Nevertheless, computing pure Nash equilibrium (if exists) of any two-player game, and computing Nash equilibrium of $2 \times n$ games (one player has two actions, and the other has $n \geq 2$ actions), are relatively simpler.

4.1 Algorithm for Computing Pure Nash Equilibria

Given a bimatrix game $(\mathbf{A}, \mathbf{B})$, suppose player 1's pure strategy of choosing action j and player 2's pure strategy of choosing action k form a pure Nash equilibrium. What does this imply?

- First, among the entries in the k-th column of $\mathbf{A}$, the j-th entry A_{jk} attains maximum.
- Second, among the entries in the j-th row of $\mathbf{B}$, the k-th entry B_{jk} attains maximum.

This gives us a simple algorithm for computing all pure Nash equilibria of the game. We give the pseudocode below, and the precise description is given in Algorithm 1.

1. For each column r of $\mathbf{A}$, circle every A_{jr} which satisfies $A_{jr} = \max_{q \in \mathcal{A}_1} A_{qr}$.
2. For each row q of $\mathbf{B}$, circle every B_{qk} which satisfies $B_{qk} = \max_{r \in \mathcal{A}_2} B_{qr}$.
3. For any (j, k) such that both A_{jk} and B_{jk} are circled, player 1's pure strategy of choosing action j and player 2's pure strategy of choosing action k form a pure Nash equilibrium.

Example 4.1. Consider the following bimatrix game, for which we want to compute all its pure Nash equilibria.

$$\begin{bmatrix} (5,4) & (3,7) & (-10,-1) & (-2,7) \\ (4,8) & (1,6) & (6,10) & (-4,8) \\ (9,6) & (3,6) & (4,3) & (-7,6) \end{bmatrix}.$$

The result after running step 1 of the algorithm is

$$\begin{bmatrix} (5,4) & (\textcircled{3},7) & (-10,-1) & (\textcircled{-2},7) \\ (4,8) & (1,6) & (\textcircled{6},10) & (-4,8) \\ (\textcircled{9},6) & (\textcircled{3},6) & (4,3) & (-7,6) \end{bmatrix}.$$

The result after running step 2 of the algorithm is

$$\begin{bmatrix} (5,4) & (\textcircled{3},\textcircled{7}) & (-10,-1) & (\textcircled{-2},\textcircled{7}) \\ (4,8) & (1,6) & (\textcircled{6},\textcircled{10}) & (-4,8) \\ (\textcircled{9},\textcircled{6}) & (\textcircled{3},\textcircled{6}) & (4,3) & (-7,\textcircled{6}) \end{bmatrix}.$$

This computes the five pure Nash equilibria of the bimatrix game, namely

- $(\ (1,0,0)\ ,\ (0,1,0,0)\ )$
- $(\ (1,0,0)\ ,\ (0,0,0,1)\ )$
- $(\ (0,1,0)\ ,\ (0,0,1,0)\ )$
- $(\ (0,0,1)\ ,\ (1,0,0,0)\ )$
- $(\ (0,0,1)\ ,\ (0,1,0,0)\ )$

Algorithm 1 Algorithm for Computing Pure Nash Equilibrium in Two-Player Game

```
for k ∈ A_2 do                                              ▷ Begin of Step 1
    z ← max_{j'∈A_1} A_{j'k}
    for j ∈ A_1 do
        if A_{jk} = z then
            Circle A_{jk}
        end if
    end for
end for

for j ∈ A_1 do                                              ▷ Begin of Step 2
    z ← max_{k'∈A_2} B_{jk'}
    for k ∈ A_2 do
        if B_{jk} = z then
            Circle B_{jk}
        end if
    end for
end for

return { (e_{1j}, e_{2k}) | (j,k) where both A_{jk}, B_{jk} are circled }   ▷ Step 3
```

4.2 Algorithm for Computing Nash Equilibria of $2 \times n$ Games

Given a bimatrix $2 \times n$ game $(\mathbf{A}, \mathbf{B})$, let $(\mathbf{x}_1, \mathbf{x}_2)$ denote a Nash equilibrium. Let $\mathsf{supp}(\mathbf{x}_i)$ denote the number of entries in $\mathbf{x}_i$ with non-zero probabilities. Using the algorithm in Section 4.1, we can compute all pure Nash equilibria, i.e., Nash equilibrium with $\mathsf{supp}(\mathbf{x}_1) = \mathsf{supp}(\mathbf{x}_2) = 1$. Next, we present an algorithm that computes all Nash equilibria where $\mathsf{supp}(\mathbf{x}_1) = \mathsf{supp}(\mathbf{x}_2) = 2$. You will find later in this course that for *almost* every $2 \times n$ game, all its Nash equilibria $(\mathbf{x}_1, \mathbf{x}_2)$ must satisfy either $\mathsf{supp}(\mathbf{x}_1) = \mathsf{supp}(\mathbf{x}_2) = 1$ or $\mathsf{supp}(\mathbf{x}_1) = \mathsf{supp}(\mathbf{x}_2) = 2$.

Suppose $\mathbf{x}_1 = (p, 1-p)$ for some $0 < p < 1$. For each action k of player 2, let $v_{2k}(p) := u_2(\mathbf{x}_1, \mathbf{e}_{2k}) = pB_{1k} + (1-p)B_{2k}$, which is an affine function of p. By Proposition 3.2, for any p, x_{2k} can be non-zero only when $v_{2k}(p) = \max_{r \in \mathcal{A}_2} v_{2r}(p)$. Conceptually, the algorithm proceeds by drawing a plot of $v_{2k}(p)$ for all k, which helps identifying the actions that player 2 might choose for different values of p. We present the pseudocode of the algorithm below:

1. Plot $v_{2k}(p)$ for each $k \in \mathcal{A}_2$ over the interval $0 \le p \le 1$. Each plot of $v_{2k}(p)$ is a line segment.
2. Identify the *upper envelope* in the plot, which is the curve of the function $\hat{v}_2(p) := \max_{k \in \mathcal{A}_2} v_{2k}(p)$.
3. Identify each point on the upper envelope which is an intersection of two line segments plotted in Step 1. Suppose the two line segments correspond to $k_1, k_2 \in \mathcal{A}_2$.
   (a) Compute the value of p at the intersection, which is the solution of

   $$pB_{1k_1} + (1-p)B_{2k_1} \;=\; pB_{1k_2} + (1-p)B_{2k_2} \; .$$

   (b) Compute q such that $qA_{1k_1} + (1-q)A_{1k_2} \;=\; qA_{2k_1} + (1-q)A_{2k_2}$.
   (c) If $0 \le q \le 1$, then

   $$((p, 1-p) \; , \; q \cdot \mathbf{e}_{2k_1} + (1-q) \cdot \mathbf{e}_{2k_2})$$

   is a Nash equilibrium.

Example 4.2. Consider the following bimatrix game.

$$\begin{bmatrix} (0,1) & (6,12) & (8,5) & (2,10) \\ (6,11) & (4,0) & (3,10) & (9,6) \end{bmatrix} .$$

We find out all pure Nash equilibria using Algorithm 1:

$$\begin{bmatrix} (0,1) & (\textcircled{6},\textcircled{12}) & (\textcircled{8},5) & (2,10) \\ (\textcircled{6},\textcircled{11}) & (4,0) & (3,10) & (\textcircled{9},6) \end{bmatrix} .$$

The two pure Nash equilibria are $((0,1),(1,0,0,0))$ and $((1,0),(0,1,0,0))$.

Next, we use the algorithm above to find out some other Nash equilibria. In Step 1, we plot the line segments; see Figure 1 (left). In Step 2, we identify the upper envelope, which is the red curve in Figure 1 (right). In the same figure, we also mark the three points on the upper envelope which are intersections of two line segments, as instructed in Step 3.

- The first point is the intersection of $v_{21}(p)$ and $v_{23}(p)$. To find out the value of p at the intersection, we solve $p + 11(1-p) = 5p + 10(1-p)$, so $p = \frac{1}{5}$. To compute q, we solve $8(1-q) = 6q + 3(1-q)$, so $q = \frac{5}{11}$. This gives us a Nash equilibrium $\left((\frac{1}{5}, \frac{4}{5}), (\frac{5}{11}, 0, \frac{6}{11}, 0)\right)$.
- The second point is the intersection of $v_{23}(p)$ and $v_{24}(p)$. To find out the value of p at the intersection, we solve $5p + 10(1-p) = 10p + 6(1-p)$, so $p = \frac{4}{9}$. To compute q, we solve $8q + 2(1-q) = 3q + 9(1-q)$, so $q = \frac{7}{12}$. This gives us another Nash equilibrium $\left((\frac{4}{9}, \frac{5}{9}), (0, 0, \frac{7}{12}, \frac{5}{12})\right)$.
- The third point is the intersection of $v_{22}(p)$ and $v_{24}(p)$. To find out the value of p at the intersection, we solve $12p = 10p + 6(1-p)$, so $p = \frac{3}{4}$. To compute q, we solve $6q + 2(1-q) = 4q + 9(1-q)$, so $q = \frac{7}{9}$. This gives us the final Nash equilibrium $\left((\frac{3}{4}, \frac{1}{4}), (0, \frac{7}{9}, 0, \frac{2}{9})\right)$.

5 Solution Concept

Nash equilibrium is a natural and well-motivated solution concept in economics and game theory. Informally speaking, a solution concept provides a benchmark on how the outcome of a game/market should be.[3] Once a solution concept is proposed, we always ask the following questions:

[3] In the rest of this course, we refer to games and markets collectively as economies.

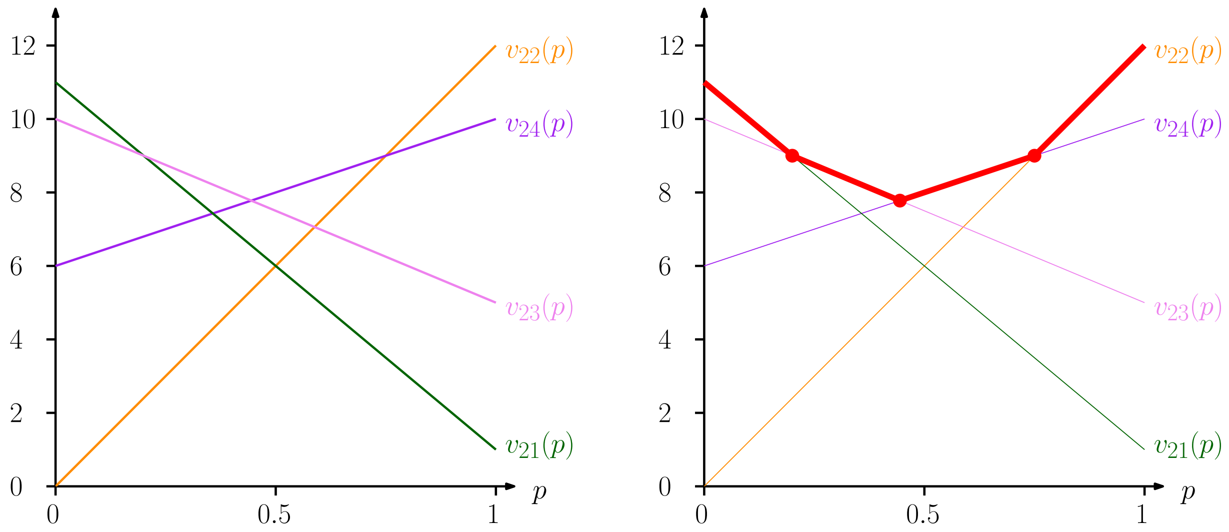

Figure 1: Plots for Example 4.2.

- **existence**: Does the solution exist for all economies, or less so, for a broad subclass of economies?
- **uniqueness**: Is there only one solution? Or is it possible to have many solutions?
- **computation**: How to efficiently compute a solution by using a computer?
- **within-the-economy computation**: In a realistic economy, players/agents typically do not possess the full information of the economy. This is in contrast with "computation" above, which assumes all economy information is part of the input to the computer. In non-full-information settings, how can the solution be achieved by the collective (distributed) behaviors of the players/agents?
- **efficiency and other measures of qualities**: As we have seen in Example 3.5, a game can have exponentially many Nash equilibria with vastly different payoffs. In general economies, is every solution efficient (in terms of yielding high overall payoffs)? In some economies, it also makes sense to ask if the solutions are *fair*.

In the rest of this courses, we will address the five types of questions in various economies.

6 Nash Equilibrium with Irrational Number Probabilities

In the examples above, the probabilities at a Nash equilibrium are rational numbers. One may wonder if the payoffs of a game are rational numbers, will it always possess a Nash equilibrium with rational number probabilities? It turns out that the answer is yes for two-player games, but not always so for games with three or more players. John Nash presented the first counterexample in his seminal paper about existence of Nash equilibrium. [1] We give another example below.

Example 6.1. In a three-player game, each player can choose an action of either 0 or 1. The three players sit in a circle. The payoffs are determined as follows; note the rotational symmetry among the three players.

- If the three players choose the same action, all players receive \$0;
- If exactly one player chooses action 1, this player receives \$1, the players on her right/left receive \$3 / \$0 respectively;
- If exactly one player chooses action 0, this player receives \$1, and the players on her right/left receive \$1 / \$0 respectively.

We claim that there is a unique Nash equilibrium $\mathbf{x}$, which occurs when each player chooses 0 with probability $1/\sqrt{2}$. First, we verify that this is indeed a Nash equilibrium. Assume players 2 and 3 sit respectively on the right and left of player 1. Then

$$\begin{aligned} u_1(\mathbf{e}_{10}, \mathbf{x}_{-1}) &= \left(\frac{1}{\sqrt{2}}\right)^2 \cdot 0 + \frac{1}{\sqrt{2}} \cdot \left(1 - \frac{1}{\sqrt{2}}\right) \cdot 3 + \left(1 - \frac{1}{\sqrt{2}}\right) \cdot \frac{1}{\sqrt{2}} \cdot 0 + \left(1 - \frac{1}{\sqrt{2}}\right)^2 \cdot 1 &= \frac{1}{\sqrt{2}} \\ u_1(\mathbf{e}_{11}, \mathbf{x}_{-1}) &= \left(\frac{1}{\sqrt{2}}\right)^2 \cdot 1 + \frac{1}{\sqrt{2}} \cdot \left(1 - \frac{1}{\sqrt{2}}\right) \cdot 0 + \left(1 - \frac{1}{\sqrt{2}}\right) \cdot \frac{1}{\sqrt{2}} \cdot 1 + \left(1 - \frac{1}{\sqrt{2}}\right)^2 \cdot 0 &= \frac{1}{\sqrt{2}} \end{aligned}$$

The BR condition of player 1 is satisfied. Due to the rotational symmetry, the BR conditions of the other two players are also satisfied. This verifies that $\mathbf{x}$ is a Nash equilibrium.

Next, we rule out the possibility of another Nash equilibrium. This is not hard but will require a few steps. We give an overview below, but leave the details to you as an exercise.

- Case 1: some player adopts a pure strategy at the Nash equilibrium.

  Without loss of generality, we assume this player is player 1.

  - If player 1 chooses action 0, then the Nash equilibrium has to be pure, at which player 2 chooses action 1 and player 3 chooses action 0. Player 1 receives \$0, but she can switch to action 1 to receive a higher payoff of \$1, a contradiction.
  - If player 1 chooses action 1, then the Nash equilibrium has to be pure, at which player 2 chooses action 0 and player 3 chooses action 1. Player 1 receives \$0, but she can switch to action 0 to receive a higher payoff of \$3, a contradiction.

- Case 2: all players use non-pure strategies at the Nash equilibrium.

  Let players 1, 2 and 3 choose action 0 with probability p, q and r respectively, where $0 < p, q, r < 1$. Due to the BR condition of each player, we have $u_i(\mathbf{e}_{i0}, \mathbf{x}_{-i}) = u_i(\mathbf{e}_{i1}, \mathbf{x}_{-i})$ for $i = 1, 2, 3$, so the following equalities hold: $1 + 2p - 2q - 2pq = 0$, $1 + 2q - 2r - 2qr = 0$ and $1 + 2r - 2p - 2rp = 0$. It is not hard to deduce that this system of equations has a unique solution of $p = q = r = \frac{1}{\sqrt{2}}$.

References

[1] John Nash. Non-cooperative games. *Annals of Mathematics*, 54(2):286–295, 1951.