Lecture 3: Existence of Nash Equilibrium

Algorithmic and Mathematical Foundations of Game Theory and Economics
Yun Kuen Cheung, School of Computing, The Australian National University

1 Existence of Nash Equilibrium

Nash equilibrium was named after the Abel and Nobel laureate[1] John Nash. However, the notion was not first proposed by him — French philosopher and mathematician Antoine Augustin Cournot applied the solution concept to analyzing competition between production firms in 1838, 90 years before Nash was born. So, why the naming? It is because Nash addressed the most fundamental question of the solution concept — if Nash equilibrium does not exist, it is *useless* no matter how naturally motivated it is. Nash proved the following theorem in his 1951 seminal paper, published in the prestigious mathematics journal, *the Annals of Mathematics.* [2]

Theorem 1.1. A *finite* normal-form game is a normal-form game with finitely many players, and each player has finitely many actions. Every such game admits at least one Nash equilibrium.

This lecture is dedicated to the proof of the above theorem. The main mathematical tool is fixed-point theorems, which provide sufficient conditions of a function (and its generalization) to guarantee the existence of a fixed point. Fixed-point theorems have been powerful tools for demonstrating the existence of many solution concepts in economics.

Why are fixed points related to Nash equilibria? In a game, suppose each player has a unique best response to any joint mixed strategy profile of the other players.[2] For any joint mixed strategy profile $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$, let $\mathrm{br}_i(\mathbf{x}_{-i})$ denote the unique best response of player i. Consider the function

$$f(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) \; := \; \left(\, \mathrm{br}_1(\mathbf{x}_{-1}) \, , \, \mathrm{br}_2(\mathbf{x}_{-2}) \, , \, \ldots, \mathrm{br}_n(\mathbf{x}_{-n}) \, \right) \, . \tag{1}$$

Observe that any fixed point of f is a Nash equilibrium by definition. In other words, proving that the game has a Nash equilibrium is equivalent to proving that the above function f has a fixed point.

2 Fixed-Point Theorems

2.1 Brouwer's Fixed-Point Theorem

We will present two fixed-point theorems. They apply to quite general topological spaces, but to keep our exposition simple, we focus on their versions on finite dimensional real spaces $\mathbb{R}^d$, which will be sufficient for the game and market applications in this course. The first theorem is due to Brouwer. While we will need a more general theorem for Nash's proof, Brouwer's theorem is a good starting point to appreciate the beauty of this kind of theorems.

Definition 2.1. Let $f : X \to X$ be a function. A fixed point of f is a point $x \in X$ such that $f(x) = x$.

Theorem 2.2. [Brouwer's Fixed-Point Theorem] Let X be a compact and convex set in $\mathbb{R}^d$. Suppose $f : X \to X$ is a continuous function. Then f admits at least one fixed point.

[1] and the protagonist of the Oscar-winning movie *A Beautiful Mind*

[2] As we have seen in the last lecture, this assumption is not correct in general. We make this assumption in the exposition to help you gain intuition.

The conditions required in Brouwer's theorem are mild. If the unique-best-response assumption made in (1) were satisfied, and if $\text{br}_i(\mathbf{x}_{-i})$ were a continuous function of $\mathbf{x}_{-i}$, our task would be complete already. However, there can be multiple best responses for certain $\mathbf{x}_{-i}$. This is why we need a generalization of Brouwer's theorem about *set-valued functions* or *correspondences*.

2.2 Kakutani's Fixed-Point Theorem

Given any set Y, 2^Y denotes the collection of all subsets of Y, i.e., $2^Y = \{Z \mid Z \subset Y\}$. A correspondence is any function of the form $\varphi : X \to 2^Y$ for any non-empty sets X, Y.

Definition 2.3. Let $\varphi : X \to 2^X$ be a correspondence. A fixed point of φ is a point $x \in X$ such that $x \in \varphi(x)$.

A generalization of continuity to correspondence is *upper hemicontinuity*, defined below.

Definition 2.4. A correspondence $\varphi : X \to 2^X$ is said to be upper hemicontinuous if for any sequences $\{x^j\}_{j=1}^{\infty}$ and $\{y^j\}_{j=1}^{\infty}$ in X such that

(i) $y^j \in \varphi(x^j)$ for any $j \geq 1$;

(ii) $\lim_{j\to\infty} x^j = x^* \in X$;

(iii) $\lim_{j\to\infty} y^j = y^* \in X$,

then $y^* \in \varphi(x^*)$.

To earn an intuition of upper hemicontinuity, we first recall the definition of continuity of a function: $f : X \to X$ is continuous if for any sequence $\{x^j\}_{j=1}^{\infty}$ in X which converges to $x^* \in X$, then $\lim_{j\to\infty} f(x^j) = f(x^*)$. Now we consider the correspondence $\varphi : X \to 2^X$ defined by $\varphi(x) = \{f(x)\}$. Continuity of f implies upper hemicontinuity of φ.

Upper hemicontinuity is less restrictive in some sense. Intuitively, it allows the correspondence to be more "inclusive" at a singular point x^*, as the following example demonstrates.

Example 2.5. Consider the normalization function f in $\mathbb{R}^2$, defined by $f(x, y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$. This function is well-defined and continuous in $\mathbb{R}^2$, except at the origin.

Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be the correspondence defined as follows. If $(x, y) \neq (0, 0)$, then $\varphi(x, y) = \{f(x, y)\}$. When $x = y = 0$, set $\varphi(0, 0) = \left\{(a, b) \in \mathbb{R}^2 \mid a^2 + b^2 = 1\right\}$, which is more "inclusive" as the set includes every possible value of $f(x, y)$ where (x, y) is in any small open neighborhood of $(0, 0)$.

It is easy to check that φ is upper hemicontinuous. Informally speaking, we may say this more inclusive correspondence *remedies* the discontinuity of f.

We are now ready to state the Kakutani's fixed-point theorem.

Theorem 2.6. [Kakutani's Fixed-Point Theorem] Let X be a compact and convex set in $\mathbb{R}^d$. Suppose $\varphi : X \to 2^X$ is an upper hemicontinuous correspondence, and $\varphi(x)$ is a non-empty and convex set for all $x \in X$. Then φ admits at least one fixed point.

The proofs of Brouwer's and Kakutani's theorems are out of the scope of this game theory course.[3] In Appendix A, we will present some counter examples to illuminate why the conditions required in the two theorems are necessary.

3 Nash's Proof

To use Kakutani's theorem for proving Nash's theorem, we need to construct a correspondence which satisfies the required conditions, such that any fixed point of the correspondence is a Nash equilibrium. Clearly, the set of joint mixed strategy profiles, $\Delta(\mathcal{A}_1)\times\Delta(\mathcal{A}_2)\times\ldots\times\Delta(\mathcal{A}_n)$, is a compact and convex set in $\mathbb{R}^d$, where $d := \sum_{i=1}^n |\mathcal{A}_i|$. Let $\mathrm{BR}_i(\mathbf{x}_{-i})$ denote the set of best responses to $\mathbf{x}_{-i}$:

$$\mathrm{BR}_i(\mathbf{x}_{-i}) \;:=\; \operatorname*{arg\,max}_{\mathbf{y}'_i\in\Delta(\mathcal{A}_i)} u_i(\mathbf{y}'_i,\mathbf{x}_{-i}) \;=\; \left\{ \mathbf{y}_i\in\Delta(\mathcal{A}_i) \;\middle|\; u_i(\mathbf{y}_i,\mathbf{x}_{-i}) = \max_{\mathbf{y}'_i\in\Delta(\mathcal{A}_i)} u_i(\mathbf{y}'_i,\mathbf{x}_{-i}) \right\}.$$

Let φ be the following correspondence:

$$\begin{aligned}\varphi(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n) \;&:=\; \mathrm{BR}_1(\mathbf{x}_{-1})\times\mathrm{BR}_2(\mathbf{x}_{-2})\times\ldots\times\mathrm{BR}_n(\mathbf{x}_{-n}) \\ &=\; \left\{(\mathbf{y}_1,\mathbf{y}_2,\ldots,\mathbf{y}_n) \;\middle|\; \forall i=1,2,\ldots,n,\; \mathbf{y}_i\in\mathrm{BR}_i(\mathbf{x}_{-i})\right\}.\end{aligned} \tag{2}$$

Example 3.1. For the two-player Rock-Paper-Scissors game, we give a few examples of $\mathrm{BR}_1(\mathbf{x}_2)$.

- If $\mathbf{x}_2 = (0,1,0)$, i.e., player 2 chooses Paper, then the best response of player 1 is to choose Scissors, i.e., $\mathrm{BR}_1(\mathbf{x}_2) = \{(0,0,1)\}$.
- If $\mathbf{x}_2 = (\frac{1}{2},\frac{1}{6},\frac{1}{3})$, then $u_1(\mathbf{e}_{1R},\mathbf{x}_2) = u_1(\mathbf{e}_{1P},\mathbf{x}_2) = \frac{1}{6}$ and $u_1(\mathbf{e}_{1S},\mathbf{x}_2) = -\frac{1}{3}$. By Proposition 2.2 in the last lecture, $\mathrm{BR}_1(\mathbf{x}_2) = \{(a,1-a,0) \mid 0\le a\le 1\}$.
- If $\mathbf{x}_2 = (\frac{1}{2}+\delta,\frac{1}{6},\frac{1}{3}-\delta)$ for a tiny positive δ, then $u_1(\mathbf{e}_{1R},\mathbf{x}_2) = \frac{1}{6}-\delta$, $u_1(\mathbf{e}_{1P},\mathbf{x}_2) = \frac{1}{6}+2\delta$ and $u_1(\mathbf{e}_{1S},\mathbf{x}_2) = -\frac{1}{3}-\delta$, and hence $\mathrm{BR}_1(\mathbf{x}_2) = \{(0,1,0)\}$.

  Observe that we only perturb $\mathbf{x}_2$ slightly, but the set $\mathrm{BR}_1(\mathbf{x}_2)$ changes significantly, so the correspondence BR_1 does not seem to match with our usual perception of continuity. However, it is upper hemicontinuous, as guaranteed by Lemma 3.3 below. Note that the set $\{(a,1-a,0) \mid 0\le a\le 1\}$ does contain $(0,1,0)$.

We point out two properties of the payoff function u_i, which will be used to show that the correspondence φ in (2) satisfies all conditions required by Kakutani's theorem. Recall that u_i is an expectation over a finite space $\mathcal{A} = \mathcal{A}_1\times\mathcal{A}_2\times\ldots\times\mathcal{A}_n$. When we expand u_i using the definition of expectation, we have

$$u_i(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n) \;=\; \sum_{(a_1,a_2,\ldots,a_n)\in\mathcal{A}} u_i(a_1,a_2,\ldots,a_n)\cdot\prod_{j=1}^n x_{j,a_j}.$$

Thus, u_i is continuous in $(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n)$. Moreover, we can write u_i as

$$u_i(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n) \;=\; \sum_{a_i\in\mathcal{A}_i}\left(\sum_{(a_1,a_2,\ldots,a_{i-1},a_{i+1},\ldots,a_n)\in\mathcal{A}_{-i}} u_i(a_1,a_2,\ldots,a_n)\cdot\prod_{\substack{j=1\\ j\neq i}}^n x_{j,a_j}\right)\cdot x_{i,a_i},$$

which indicates that for any fixed $\mathbf{x}_{-i}$, u_i is a linear function of $\mathbf{x}_i$, and hence it is concave in $\mathbf{x}_i$.

[3] For Brouwer's theorem in real spaces, there is a proof via a combinatorial result called *Sperner's lemma*, which is relatively elementary when compared to other existing proofs of the theorem. The proof is available at [1].

Proposition 3.2. For any $\mathbf{x} \in X$, $\varphi(\mathbf{x})$ is a non-empty and convex set.

Proof Sketch: It suffices to show that $\mathrm{BR}_i(\mathbf{x}_{-i})$ is a non-empty and convex set for any i, which is true since $\mathrm{BR}_i(\mathbf{x}_{-i}) = \arg\max_{\mathbf{y}'_i \in \Delta(\mathcal{A}_i)} u_i(\mathbf{y}'_i, \mathbf{x}_{-i})$, is the arg max of a continuous and concave function in the compact domain $\Delta(\mathcal{A}_i)$. □

Lemma 3.3. The correspondence φ in (2) is upper hemicontinuous.

Proof: Let $X = \Delta(\mathcal{A}_1) \times \Delta(\mathcal{A}_2) \times \ldots \times \Delta(\mathcal{A}_n)$. Let

$$\left\{\mathbf{x}^j = (\mathbf{x}_1^j, \mathbf{x}_2^j, \ldots, \mathbf{x}_n^j)\right\}_{j=1}^{n} \quad , \quad \left\{\mathbf{y}^j = (\mathbf{y}_1^j, \mathbf{y}_2^j, \ldots, \mathbf{y}_n^j)\right\}_{j=1}^{n}$$

be any two sequences in X with limits $\mathbf{x}^*, \mathbf{y}^* \in X$ respectively, and $\mathbf{y}^j \in \varphi(\mathbf{x}^j)$ for any $j \geq 1$. To prove the lemma, it suffices to show that $\mathbf{y}_i^* \in \mathrm{BR}_i(\mathbf{x}_{-i}^*)$ for every i.

Suppose the contrary, i.e., there exists i such that $\mathbf{y}_i^* \notin \mathrm{BR}_i(\mathbf{x}_{-i}^*)$. Then by Proposition 3.2, there exists $\mathbf{z}_i \in \mathrm{BR}_i(\mathbf{x}_{-i}^*)$ such that $u_i(\mathbf{z}_i, \mathbf{x}_{-i}^*) > u_i(\mathbf{y}_i^*, \mathbf{x}_{-i}^*)$. Since the payoff function u_i is continuous,

$$\lim_{j\to\infty} u_i(\mathbf{z}_i, \mathbf{x}_{-i}^j) \;=\; u_i(\mathbf{z}_i, \mathbf{x}_{-i}^*) \;>\; u_i(\mathbf{y}_i^*, \mathbf{x}_{-i}^*) \;=\; \lim_{j\to\infty} u_i(\mathbf{y}_i^j, \mathbf{x}_{-i}^j).$$

Thus, there exists a sufficiently large j such that $u_i(\mathbf{z}_i, \mathbf{x}_{-i}^j) > u_i(\mathbf{y}_i^j, \mathbf{x}_{-i}^j)$. But since $\mathbf{y}^j \in \varphi(\mathbf{x}^j)$, we have $\mathbf{y}_i^j \in \mathrm{BR}_i(\mathbf{x}_{-i}^j)$ and hence $u_i(\mathbf{z}_i, \mathbf{x}_{-i}^j) \leq u_i(\mathbf{y}_i^j, \mathbf{x}_{-i}^j)$, a contradiction. □

To complete the proof of Nash's theorem, Theorem 2.6, Proposition 3.2 and Lemma 3.3 together imply that φ has a fixed point, say it is $(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$. Directly from the definition of φ and BR_i, for each player i, $\mathbf{x}_i$ maximizes $u_i(\mathbf{x}_i, \mathbf{x}_{-i})$. Thus, the fixed point is a Nash equilibrium.

4 Discussions on Nash's Theorem

Observe that Proposition 3.2 and Lemma 3.3 are valid as long as for each player i, (i) $u_i : X \to \mathbb{R}$ is continuous, and (ii) $u_i(\mathbf{x}_i, \mathbf{x}_{-i})$ is quasi-concave in $\mathbf{x}_i$.[4] Thus, Nash's theorem is true for games far beyond normal-form games.

We present a game with no Nash equilibrium below, in which condition (ii) does not hold.

Example 4.1. In a two-player zero-sum game, each player i chooses $x_i \in [0, 1]$, and $u_1(x_1, x_2) = -u_2(x_1, x_2) = |x_1 - x_2|$. Note that u_1 is not quasi-concave in x_1, but u_2 is quasi-concave in x_2. Suppose the contrary that a Nash equilibrium (x_1^*, x_2^*) exists. Since player 2 wants to minimize the distance, $x_2^* = x_1^*$. But player 1 wants to maximize the distance, so $x_1^* \neq x_2^*$, a contradiction.

Since there is no Nash equilibrium, some condition required in Kakutani's theorem must not hold; the violated condition is the convexity of $\varphi(x_1, x_2)$. Recall that $\varphi(x_1, x_2) = \mathrm{BR}_1(x_2) \times \mathrm{BR}_2(x_1)$. We compute the two best response correspondences below:

$$\mathrm{BR}_1(x_2) = \begin{cases} \{1\}, & \text{if } x_2 < 0.5; \\ \{0\}, & \text{if } x_2 > 0.5; \\ \{0, 1\}, & \text{if } x_2 = 0.5. \end{cases} \qquad\qquad \mathrm{BR}_2(x_1) = \{x_1\}\ .$$

Observe that $\varphi(x_1, x_2)$ is convex for almost any $(x_1, x_2) \in [0, 1] \times [0, 1]$, except when $x_2 = 0.5$.

[4] Let X be a convex set in $\mathbb{R}^n$. A function $u_i : X \to \mathbb{R}$ is quasi-concave on X if for all $x, x' \in X$ and for all $\lambda \in [0, 1]$, we have $f(\lambda x + (1 - \lambda)x') \geq \min\{f(x), f(x')\}$. It is not hard to see that if u_i is concave then u_i is quasi-concave, but not vice versa.

Nash's theorem applies for finite games. What about infinite games, with either infinitely many players, or infinitely many actions per player? We present a game with no Nash equilibrium below, in which there are two players but each player has infinitely many actions.

Example 4.2. In a two-player zero-sum game with $\mathcal{A}_1 = \mathcal{A}_2 = \mathbb{N}$, each player i chooses $a_i \in \mathbb{N}$. $u_1(a_1, a_2) = 1$ if $a_1 = a_2$, otherwise $u_1(a_1, a_2) = 0$. Suppose the contrary that a Nash equilibrium $(\mathbf{x}_1^*, \mathbf{x}_2^*) \in \Delta(\mathbb{N}) \times \Delta(\mathbb{N})$ exists. Player 2 wants to avoid choosing the same action as player 1. It is not hard to see that for any $\mathbf{x}_1^* \in \Delta(\mathbb{N})$, $\sup_{\mathbf{x}_2 \in \Delta(\mathbb{N})} u_2(\mathbf{x}_1^*, \mathbf{x}_2) = 0$. Thus, $u_2(\mathbf{x}_1^*, \mathbf{x}_2^*) = 0$. On the other hand, player 1 wants to match with the action of player 2. Given $\mathbf{x}_2^* \in \Delta(\mathbb{N})$, player 1 may use a pure strategy of choosing an action $j^* \in \arg\max_{j \in \mathbb{N}} x_{2j}$. Denote this pure strategy by $\mathbf{e}_{j^*}$. Note that $u_1(\mathbf{x}_1^*, \mathbf{x}_2^*) \geq u_1(\mathbf{e}_{j^*}, \mathbf{x}_2^*) = x_{2j^*} > 0$. Thus, $u_1(\mathbf{x}_1^*, \mathbf{x}_2^*) + u_2(\mathbf{x}_1^*, \mathbf{x}_2^*) > 0$, a contradiction to the fact that the game is zero-sum.

In Appendix B, we will present another game with no Nash equilibrium, in which there are infinitely many players but each player has only two action.

5 Existence vs. Computation

At the beginning of this note, I wrote "if a solution concept does not exist, it is useless no matter how naturally motivated it is". Nash's theorem guarantees existence of Nash equilibrium, marvelous! Nevertheless, the *usefulness* of the solution concept is not imminent yet. Let me modify the quote:

If a solution concept cannot be found efficiently,
it is useless even it exists and is naturally motivated.

In computer science language, the above quote can be rephrased as

A solution concept is useful not only because it always exists,
but also because it can be computed by an algorithm efficiently.

Fixed-point theorems are powerful and beautiful mathematical tools, but they have their limitation: their proofs are typically *non-constructive*, i.e., they do not shed much insight on where a fixed point locates. We do need new ideas to compute Nash equilibrium. It turns out that devising (efficient) algorithms to compute Nash equilibrium, even just for two-player games, is not easy. In the next few lectures, we focus on two-player games and present algorithms for computing Nash equilibrium.

References

[1] Jacob Fox. Lecture 3: Sperner's lemma and Brouwer's theorem. Click here to access the pdf file.

[2] John Nash. Non-cooperative games. *Annals of Mathematics*, 54(2):286–295, 1951.

A Required Conditions in the Fixed-Point Theorems

			$S \times S$		f
S	$\mathrm{br}_1(s_2)$	$\mathrm{br}_2(s_1)$	compact?	convex?	continuous?
interval $[0, 1]$	$g(s_2)$	$g(s_1)$	Yes	Yes	No
$\left\{(s_{i1}, s_{i2}) \in \mathbb{R}^2 \mid \frac{1}{2} \leq (s_{i1})^2 + (s_{i2})^2 \leq 1\right\}$	$(-s_{21}, s_{22})$	$(s_{11}, -s_{12})$	Yes	No	Yes
$\mathbb{R}$	$s_2 + 1$	$s_1 + 1$	No	Yes	Yes

In the table above, we present three two-player games which do not have Nash equilibrium. In each game, both players have identical strategy spaces, denoted by S. We denote player i's choice of strategy by s_i. For each player i, we specify the unique best response $\mathrm{br}(s_{3-i})$ in the table. Her payoff function is $u_i(s_i, s_{3-i}) = -\|s_i - \mathrm{br}(s_{3-i})\|$. Since the best responses are unique, we can consider the best response function $f : S \times S \to S \times S$ in (1), instead of the correspondence (2).

We leave to you as an exercise to show that each f has no fixed point in its respective domain $S \times S$. Thus, in each case, at least one of the following three conditions required in Brouwer's theorem is violated: (i) $S \times S$ is compact, (ii) $S \times S$ is convex, (iii) f is continuous. In each game, exactly one distinct condition from (i), (ii), (iii) is violated. This demonstrates that each condition is (somewhat) necessary.[5]

In the first game, the function g is $g(s) = (s + 0.1) \mod 1$. Note that g is not continuous. If you are not familiar with the "mod" notation, below is a more elementary definition:

$$g(t) \;=\; \begin{cases} t + 0.1, & \text{if } 0 \leq t < 0.9 \;; \\ t - 0.9, & \text{if } 0.9 \leq t \leq 1 \;. \end{cases}$$

The conditions required in Kakutani's theorem (Theorem 2.6) are similar, but to deal with correspondences it requires one more condition that $\varphi(x)$ is a non-empty and convex set for all $x \in X$. In Example 4.1, we also demonstrated that the convexity condition is necessary. The non-empty condition is also necessary: if $\varphi(x)$ is the empty set for all $x \in X$, the correspondence is trivially upper hemicontinuous, but it admits no fixed point.

B A Game with Infinitely Many Players

Here, we present a game with infinitely many players and each player has finitely many actions, and the game has no Nash equilibrium. The players are indexed by positive integers in $\mathbb{N}$. Each player has two actions, H (head) and T (tail). For any joint action profile $\mathbf{a} = \{a_j\}_{j \in \mathbb{N}}$, let $C_H(\mathbf{a}) = |\{j | a_j = H\}|$ and $C_T(\mathbf{a}) = |\{j | a_j = T\}|$. For any player i, let $C_H^i(\mathbf{a}) = |\{j \neq i | a_j = H\}|$ and $C_T^i(\mathbf{a}) = |\{j \neq i | a_j = H\}|$. We make the following simple observation.

Observation 1. For any player i, $C_H(\mathbf{a}) = \infty$ if and only if $C_H^i(\mathbf{a}) = \infty$, and $C_T(\mathbf{a}) = \infty$ if and only if $C_T^i(\mathbf{a}) = \infty$.

For each player i, the payoff function u_i is defined as follows:

$$u_i(a_i, \mathbf{a}_{-i}) \;=\; \begin{cases} 0, & \text{if } a_i = H,\; C_H^i = \infty,\; C_T^i < \infty \\ 1, & \text{if } a_i = T,\; C_H^i = \infty,\; C_T^i < \infty \\ 1, & \text{if } a_i = H,\; C_H^i < \infty,\; C_T^i = \infty \\ 0, & \text{if } a_i = T,\; C_H^i < \infty,\; C_T^i = \infty \\ i, & \text{if } a_i = H,\; C_H^i = \infty,\; C_T^i = \infty \\ 0, & \text{if } a_i = T,\; C_H^i = \infty,\; C_T^i = \infty \end{cases}$$

[5] In Brouwer's theorem (Theorem 2.2), the condition "X is convex" is not strictly necessary. It can be replaced by "X is homeomorphic to a closed ball", or in more layman term, X does not have a *hole* in the middle.

Suppose the contrary that there exists a Nash equilibrium $\{\mathbf{x}_i\}_{i\in\mathbb{N}}$, where $\mathbf{x}_i$ is the mixed strategy of player i at the equilibrium. $\{\mathbf{x}_i\}_{i\in\mathbb{N}}$ determines a probability measure over $\{H,T\}^{\mathbb{N}}$. Under this probability measure, let

$$\begin{aligned} p &:= \mathbb{P}\left[C_H = \infty,\ C_T < \infty\right] \\ q &:= \mathbb{P}\left[C_H < \infty,\ C_T = \infty\right] \\ r &:= \mathbb{P}\left[C_H = \infty,\ C_T = \infty\right] \end{aligned}$$

Note that $p+q+r=1$. By Observation 1, $u_i(H,\mathbf{x}_{-i}) = q+ri$ and $u_i(T,\mathbf{x}_{-i}) = p$.

We claim that $r=0$. Suppose the contrary that $r>0$. Then every player i with $i > \frac{p-q}{r}$ must choose H with probability 1 at the Nash equilibrium, but this implies $C_T < \infty$ almost surely, which forces $r=0$, a contradiction.

We need the following well-known lemma in probability theory to proceed.

Lemma B.1. [Second Borel-Cantelli Lemma] Let $\{E_j\}_{j\in\mathbb{N}}$ be a sequence of independent events. If $\sum_{j\in\mathbb{N}} \mathbb{P}\left[E_j\right] = \infty$, then the event that E_j occurs infinitely often has probability 1.

We claim that either $p=1$ or $q=1$. Suppose the contrary, i.e., $0<p,q<1$. By the contrapositive of the second Borel-Cantelli lemma, $\sum_{j\in\mathbb{N}} x_{j,H} < \infty$ and $\sum_{j\in\mathbb{N}} x_{j,T} < \infty$, but this is impossible since $\sum_{j\in\mathbb{N}}(x_{j,H}+x_{j,T}) = \sum_{j\in\mathbb{N}} 1 = \infty$.

If $p=1$ and $q=0$, then at the Nash equilibrium, each player's best response is to choose T with probability 1. But this implies $q=1$, a contradiction.

Analogously, if $p=0$ and $q=1$, then at the Nash equilibrium, each player's best response is to choose H with probability 1. But this implies $p=1$, a contradiction.