# L4: More Applications of Labelled Constructions and Boxed Product

**Analytic Combinatorics**
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In the previous lecture, we discussed the foundational labelled constructions: disjoint union, labelled product, sequence, set and cycle. We now present more applications of these constructions, followed by the last labelled construction, the boxed product.

## Labelled Trees, Graphs and Mappings

The labelled constructions become especially effective when a combinatorial object decomposes into naturally labelled components. Rooted trees are built from a root and a set of subtrees; sparse graph classes often decompose into cycles decorated by trees; mappings are functional digraphs whose connected components are cycles of rooted trees. The examples below show how these structural decompositions translate directly into EGFs.

**Example 1.** Let $\mathcal{T}$ be the class of *rooted labelled trees*. A rooted labelled tree consists of a single root node (an atom $\mathcal{Z}$) connected to an unordered collection (a SET) of subtrees, each of which is itself a rooted labelled tree. Because the labels of the tree must be partitioned among the root and the subtrees, this structural definition translates to the symbolic specification:

$$\mathcal{T} \;=\; \mathcal{Z} \star \mathrm{SET}(\mathcal{T}).$$

By the labelled product and set theorems, this immediately yields the functional equation

$$T(z) \;=\; z\exp(T(z)).$$

While we cannot extract the exact coefficients of $T(z)$ using elementary series expansions alone, advanced tools (which we will study in later lectures) can be used to show that $[z^n]T(z) = \frac{n^{n-1}}{n!}$. Thus, the number of rooted labelled trees on $n$ vertices is $t_n = n^{n-1}$.

The functional equation also yields a recurrence. Differentiating $T(z) = z\exp(T(z))$ gives

$$T'(z) \;=\; \exp(T(z)) + z\exp(T(z))T'(z)$$

and hence

$$zT'(z) \;=\; z\exp(T(z)) + z^2\exp(T(z))T'(z) \;=\; T(z) + zT'(z)T(z),$$

which is a new functional equation avoiding the term $\exp(T(z))$. Writing $T(z) = \sum_{n\geq 1} t_n z^n/n!$ and comparing the coefficient of $z^n/n!$ on both sides, we obtain

$$(n-1)t_n \;=\; \sum_{k=1}^{n-1} \binom{n}{k} k\, t_k t_{n-k}$$

for $n \geq 2$, with initial condition $t_1 = 1$. Combinatorially, consider a rooted tree with a marked non-root vertex $u$, and let $p$ be the parent of $u$. Cutting the edge $\{p, u\}$ produces a rooted component containing the original root with marked vertex $p$, together with a second rooted component whose root is $u$. The $\binom{n}{k}$ factor chooses the labels of the first component. The first component is enumerated via $kt_k$, while the second component is enumerated via $t_{n-k}$.

**Example 2.** An *unrooted labelled tree* lacks a designated root. Since any unrooted tree on $n$ vertices can be uniquely rooted by choosing one of its $n$ vertices, the number of unrooted labelled trees is exactly $t_n/n = n^{n-2}$. This famous result is known as *Cayley's Formula*.

Let $\mathcal{U}$ be the class of unrooted labelled trees. While we just related their counting sequences via Cayley's formula, we can also derive their EGF $U(z)$ directly from the structure of trees using a combinatorial argument. For any finite tree, the number of vertices $V$ and the number of edges $E$ always satisfy $V - E = 1$. Consequently, the number of vertex-rooted versions of a tree minus the number of edge-rooted versions is exactly 1, so each unrooted tree is counted exactly once by this difference.

Combinatorially, choosing a distinguished vertex in an unrooted tree is equivalent to rooting the tree, yielding the class of rooted trees $\mathcal{T}$. Choosing a distinguished edge is equivalent to selecting an unordered pair of rooted trees (since removing an edge partitions the tree into two rooted subtrees). The class of such edge-rooted trees is an unordered pair, which is $\mathrm{SET}_2(\mathcal{T})$. Therefore, the class of unrooted trees can be formally expressed as the difference:

$$\mathcal{U} \ = \ \mathcal{T} - \mathrm{SET}_2(\mathcal{T}).$$

Translating this into EGF, we obtain:

$$U(z) \ = \ T(z) - \frac{T(z)^2}{2}. \tag{1}$$

**Exercise 3.** In the previous example, we derived the EGF for unrooted labelled trees combinatorially. We can also derive Equation (1) purely algebraically by exploiting the relationship between the coefficients of $U(z)$ and $T(z)$.

(a) Since an unrooted tree on $n$ vertices can be uniquely rooted by choosing any of its $n$ vertices, we have $t_n = nu_n$. Use this equality to show that $U(z) = \int_0^z \frac{T(w)}{w} \mathsf{d}w$.

(b) Recall from Example 1 that $T(w) = w \exp(T(w))$. Use this relation and perform the change of variables $y = T(w)$ to show that $\mathsf{d}w = e^{-y}(1-y)\mathsf{d}y$ and derive Equation (1).

**Example 4.** We count rooted labelled trees of bounded height. Let $\mathcal{H}_{\leq h}$ denote the class of rooted labelled trees of height at most $h$, where a single root has height 0. Then $H_{\leq 0}(z) = z$.

For $h \geq 0$, a tree of height at most $h+1$ consists of a root attached to an unordered set of subtrees of height at most $h$. Therefore

$$\mathcal{H}_{\leq h+1} \ = \ \mathcal{Z} \star \mathrm{SET}(\mathcal{H}_{\leq h}), \quad \text{and hence} \quad H_{\leq h+1}(z) \ = \ z \exp(H_{\leq h}(z)).$$

Iterating this recurrence gives

$$H_{\leq 1}(z) \ = \ ze^z; \qquad H_{\leq 2}(z) \ = \ z\exp(ze^z); \qquad H_{\leq 3}(z) \ = \ z\exp(z\exp(ze^z))\,.$$

It is also useful to isolate trees of exact height. For $h \geq 1$, we have

$$H_{=h}(z) \ = \ H_{\leq h}(z) - H_{\leq h-1}(z).$$

**Example 5.** A *rooted Cayley tree with odd degrees* is a rooted labelled tree in which every vertex has odd degree. The root is the only vertex whose degree is exactly its number of children; every other vertex has one additional parent edge.

Let $\mathcal{R}$ be the auxiliary class of rooted labelled trees in which the root has even degree and every other vertex has odd degree. In such a tree, the root has an even number of children. Each child subtree is again an object of $\mathcal{R}$: the child has an even number of children in its own subtree, and the parent edge then makes its degree odd in the larger tree. Therefore

$$\mathcal{R} \;=\; \mathcal{Z} \star \mathrm{SET}_{\mathrm{even}}(\mathcal{R}).$$

The even set construction contributes $\sum_{j\geq 0} R(z)^{2j}/(2j)! = \cosh R(z)$, so

$$R(z) \;=\; z \cosh R(z).$$

Let $\mathcal{O}$ be the desired class of rooted labelled trees in which every vertex has odd degree. The root must now have an odd number of children, and each child subtree belongs to $\mathcal{R}$. Hence

$$\mathcal{O} \;=\; \mathcal{Z} \star \mathrm{SET}_{\mathrm{odd}}(\mathcal{R}).$$

Since the odd set construction contributes $\sum_{j\geq 0} R(z)^{2j+1}/(2j+1)! = \sinh R(z)$, the EGF for rooted Cayley trees with odd degrees is

$$O(z) \;=\; z \sinh R(z).$$

**Exercise 6.** Let $\mathcal{V}$ be the class of rooted Cayley trees in which every internal node has even degree, and let $V(z)$ be its EGF. Here a root of degree 1 is a leaf, not an internal node, and is therefore allowed. Derive the following functional equations which characterize $V(z)$ via an EGF $Q(z)$ of an auxiliary combinatorial class.

$$Q(z) \;=\; z\,(1 + \sinh Q(z)) \qquad\qquad V(z) \;=\; z\,(Q(z) + \cosh Q(z))\,.$$

Using these functional equations, compute $v_j$ for $1 \leq j \leq 9$.

**Example 7.** A *2-regular graph* is a simple graph in which every vertex has degree exactly 2. Any such graph is composed of a set of disjoint undirected cycles. Since the graph is simple, cycles of length 1 (self-loops) and length 2 (multiple edges) are forbidden.

An undirected cycle of length at least 3 has a symmetry factor of $\frac{1}{2}$ relative to a directed cycle, because traversing the sequence backwards yields the same undirected cycle. Let $\mathcal{K}$ be the quotient of directed cycles of length at least 3 by reversal. Its EGF is

$$K(z) \;=\; \frac{1}{2}\left(\log \frac{1}{1-z} - z - \frac{z^2}{2}\right).$$

The class of 2-regular graphs is $\mathcal{R} = \mathrm{SET}(\mathcal{K})$, and hence the EGF for 2-regular graphs is

$$R(z) \;=\; \exp(K(z)) \;=\; \exp\left(\frac{1}{2}\log\frac{1}{1-z} - \frac{z}{2} - \frac{z^2}{4}\right) \;=\; \frac{e^{-z/2-z^2/4}}{\sqrt{1-z}}.$$

**Example 8.** The *excess* of a connected graph is defined as the number of edges minus the number of vertices, $k = E - V$. Trees have an excess of $-1$.

A connected graph with excess 0 is called a *unicyclic graph*, as it contains exactly one cycle. Structurally, a unicyclic graph can be viewed as a cycle of rooted labelled trees. However, since the cycle in a standard simple graph is *undirected*, we must be careful with symmetries.

A directed cycle of rooted trees corresponds to $\mathrm{CYC}(\mathcal{T})$. For a cycle of length $\geq 3$, an undirected cycle can be traversed in exactly two directions, meaning there are exactly half as many undirected cycles as directed cycles. Cycles of length 1 or 2 are not allowed in simple graphs. Therefore, the class of unicyclic graphs $\mathcal{G}_0$ is:

$$\mathcal{G}_0 \;=\; \frac{1}{2}\mathrm{CYC}_{\geq 3}(\mathcal{T}).$$

Translating to generating functions yields:

$$G_0(z) \;=\; \frac{1}{2}\left(\log\frac{1}{1-T(z)} - T(z) - \frac{T(z)^2}{2}\right).$$

**Example 9.** A mapping or function $f : \{1,\ldots,n\} \to \{1,\ldots,n\}$ can be naturally viewed as a directed functional graph on $n$ vertices, where a directed edge exists from $i$ to $f(i)$. Because every vertex has out-degree exactly 1, the graph consists of a set of connected components. Each component is a directed cycle of nodes, and each node on the cycle is the root of a directed tree whose edges point *towards* the cycle. Symbolically, such a structure is a *cycle of rooted trees*. The class of all mappings is therefore a set of these components:

$$\mathcal{M} \;=\; \mathrm{SET}(\mathrm{CYC}(\mathcal{T})).$$

Applying the set and cycle translations, the EGF for mappings is:

$$M(z) \;=\; \exp\left(\log\frac{1}{1-T(z)}\right) \;=\; \frac{1}{1-T(z)}.$$

Extracting the coefficients of $M(z)$ algebraically is non-trivial, but we already know the answer combinatorially. There are exactly $n^n$ functions from $\{1,\ldots,n\}$ to itself for $n \geq 1$, while the empty mapping contributes the constant term. Thus, $M(z) = 1 + \sum_{n\geq 1} n^n \frac{z^n}{n!}$.

**Example 10.** In this example, we derive the EGF $M_{\leq 2}(z)$ for the class of mappings $f : \{1,\ldots,n\} \to \{1,\ldots,n\}$ such that

$$|\{f(f(f(i))) \mid i \in \{1,\ldots,n\}\}| \;\leq\; 2.$$

Let $S = f^3(\{1,\ldots,n\})$. Then $S$ is forward-invariant under $f$, meaning that $f(S) \subseteq S$: if $s = f^3(i)$, then $f(s) = f^4(i) = f^3(f(i)) \in S$. Moreover, every vertex outside $S$ reaches $S$ after at most three applications of $f$. Thus each element of $S$ is a core vertex with an attached rooted tree of height at most 3, as studied in Example 4. We will use the EGFs $H_{\leq h}(z)$ and $H_{=h}(z)$ derived there.

The case $|S| = 0$ contributes only the empty mapping. For the case $|S| = 1$, the unique core vertex must be fixed by $f$, and all other vertices lie in a rooted tree of height at most 3 attached to it. The two cases contribute

$$1 + H_{\leq 3}(z)$$

to the EGF $M_{\le 2}(z)$. Now consider the case $|S| = 2$. Let $S = \{s_1, s_2\}$. There are two types of maps on the two core vertices. First, if the core map is a bijection, it is either the identity ($f(s_i) = s_i$) or a transposition ($f(s_i) = s_{3-i}$). In both cases, the two core vertices both occur in $f^3(\{1, \ldots, n\})$ automatically, and each core vertex carries an arbitrary height-at-most-3 rooted tree. The two bijections contribute

$$2 \cdot \frac{H_{\le 3}(z)^2}{2!} \;=\; H_{\le 3}(z)^2$$

to the EGF $M_{\le 2}(z)$. Second, the core map has one fixed vertex and one transient vertex pointing to it, i.e., $f(s_1) = f(s_2) = s_1$. The fixed core vertex carries an arbitrary height-at-most-3 tree. However, the transient core vertex appears in $f^3(\{1, \ldots, n\})$ only if some vertex maps to it after exactly three applications of $f$. Thus, its attached tree must have height *exactly* 3. This contributes

$$H_{\le 3}(z) H_{=3}(z) \;=\; H_{\le 3}(z)(H_{\le 3}(z) - H_{\le 2}(z))$$

to the EGF $M_{\le 2}(z)$. Combining all the above contributions yields

$$\begin{aligned} M_{\le 2}(z) &= 1 + H_{\le 3}(z) + 2H_{\le 3}(z)^2 - H_{\le 2}(z)H_{\le 3}(z) \\ &= 1 + z\exp(z\exp(ze^z)) + 2z^2\exp(2z\exp(ze^z)) - z^2\exp(ze^z + z\exp(ze^z)). \end{aligned}$$

**Exercise 11.** Let $M_{=3}(z)$ be the EGF for mappings $f : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that $|\{f(f(f(i))) \mid i \in \{1, \ldots, n\}\}| = 3$. Show that

$$M_{=3}(z) \;=\; H_{\le 3}(z)^3 + 3H_{\le 3}(z)^2 H_{=3}(z) + \frac{1}{2} H_{\le 3}(z) H_{=3}(z)^2.$$

A systematic way of deriving the above EGF is to consider the cases $|f(S)| = 1, 2, 3$ and their subcases, where $S = f^3(\{1, \ldots, n\})$.

## Boxed Product

Finally, we consider a class of problems where we must restrict which component receives the smallest (or largest) label.

**Definition 12.** The boxed product (or min-labelled product) of two labelled classes $\mathcal{A}$ and $\mathcal{B}$, denoted $\mathcal{A}^{\Box} \star \mathcal{B}$, is the subset of the standard labelled product $\mathcal{A} \star \mathcal{B}$ where the globally *smallest* label in the combined object is rigidly constrained to belong to the $\mathcal{A}$ component.

**Theorem 13.** If $\mathcal{C} = \mathcal{A}^{\Box} \star \mathcal{B}$, the EGF of $\mathcal{C}$ is

$$C(z) \;=\; \int_0^z A'(t)B(t)\,\mathsf{d}t.$$

Equivalently, this can be written as the differential equation $C'(z) = A'(z)B(z)$ with initial condition $C(0) = 0$.

**Proof:** To form an object of size $n$, the globally smallest label (which is 1) must be placed in the $\mathcal{A}$ component. Let the size of the $\mathcal{A}$ component be $k$. The remaining $k-1$ labels for the $\mathcal{A}$ component must be chosen from the available $n-1$ labels (since 1 is already taken). The $\mathcal{B}$ component receives the other $n-k$ labels. Thus, the number of ways is:

$$c_n \;=\; \sum_{k=1}^{n} \binom{n-1}{k-1} a_k b_{n-k}.$$

Dividing by $n!$:

$$\frac{c_n}{n!} \;=\; \frac{1}{n}\sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!}\frac{a_k}{(n-1)!}b_{n-k} \;=\; \frac{1}{n}\sum_{k=1}^{n} \frac{a_k}{(k-1)!}\frac{b_{n-k}}{(n-k)!}.$$

We recognize that the sum $\sum_{k=1}^{n} \frac{a_k}{(k-1)!}\frac{b_{n-k}}{(n-k)!}$ is the coefficient of $z^{n-1}$ in the product of the power series $A'(z)$ and $B(z)$. Therefore, $\frac{c_n}{n!} = \frac{1}{n}[z^{n-1}]A'(z)B(z)$. Since dividing the coefficient of $z^{n-1}$ by $n$ corresponds to formally integrating the power series from 0 to $z$, we conclude that $C(z) = \int_0^z A'(t)B(t)\,\mathsf{d}t$. □

**Example 14.** The boxed product provides an elegant alternative way to derive the EGF for the cycle construction. A directed cycle can be viewed as a sequence of atoms where cyclic shifts are considered equivalent. We can break this symmetry by identifying a unique canonical representation for every cycle: we simply cut the cycle at the atom with the globally smallest label and lay it out linearly.

Thus, every cycle has a unique representation as a sequence of atoms in which the first atom holds the globally smallest label. Using the boxed product, the class of cycles of atoms is:

$$\mathcal{C} \;=\; \mathcal{Z}^{\square} \star \mathrm{SEQ}(\mathcal{Z}).$$

Let $Z(z) = z$ and $S(z) = \frac{1}{1-z}$. By Theorem 13, we have:

$$C(z) \;=\; \int_0^z Z'(t)S(t)\,\mathsf{d}t \;=\; \int_0^z 1\cdot\frac{1}{1-t}\,\mathsf{d}t \;=\; \log\frac{1}{1-z}.$$

This perfectly matches the result derived in the previous lecture, showcasing the versatility of the symbolic method.

**Example 15.** An *increasing tree* is a rooted labelled tree where the sequence of labels along any path from the root to a leaf is strictly increasing. This condition strictly implies that the root node *must* contain the globally smallest label in the entire tree.

Let $\mathcal{I}$ be the class of increasing trees. The tree consists of a root node (class $\mathcal{Z}$) attached to a set of subtrees (class $\mathrm{SET}(\mathcal{I})$). However, because the root must hold the smallest label, we use the boxed product, boxing the root:

$$\mathcal{I} \;=\; \mathcal{Z}^{\square} \star \mathrm{SET}(\mathcal{I}).$$

Let $Z(z) = z$. Then $Z'(z) = 1$. Applying Theorem 13:

$$I(z) \;=\; \int_0^z Z'(t)\exp(I(t))\,\mathsf{d}t \;=\; \int_0^z \exp(I(t))\,\mathsf{d}t.$$

Differentiating both sides yields the autonomous differential equation $I'(z) = \exp(I(z))$ with the initial condition $I(0) = 0$. Separating variables gives $e^{-I}dI = dz$. Integrating yields $-e^{-I} = z + C$. Since $I(0) = 0$, $C = -1$, so $e^{-I(z)} = 1 - z$, which means:

$$I(z) \;=\; \log\frac{1}{1-z},$$

which is the EGF of $\mathrm{CYC}(\mathcal{Z})$. Thus, there are $(n-1)!$ increasing trees on $n$ vertices.

**Example 16.** An *increasing binary tree* is an increasing tree where every node has either zero, one (left or right), or two children. Since the left and right children are distinguished, a node is attached to a sequence of exactly two optional subtrees. The optional subtree class is $\mathcal{E} + \mathcal{B}$.

Boxing the root, the specification is $\mathcal{B} = \mathcal{Z}^{\square} \star (\mathcal{E} + \mathcal{B})^2$. This translates to the integral equation:

$$B(z) \;=\; \int_0^z (1 + B(t))^2 \, \mathsf{d}t.$$

Differentiating yields the separable differential equation $B'(z) = (1 + B(z))^2$ with $B(0) = 0$.

$$\frac{\mathsf{d}B}{(1+B)^2} \;=\; \mathsf{d}z \implies -\frac{1}{1+B} \;=\; z + C.$$

Using $B(0) = 0$, we find $C = -1$, yielding $B(z) = \frac{z}{1-z}$. Thus, there are $n![z^n]B(z) = n!$ binary increasing trees.

**Exercise 17.** Consider *increasing complete binary trees*, where every node has either zero or two children.

(a) Suppose that the left and right children are distinguished. Show that the specification is

$$\mathcal{B}_1 \;=\; \mathcal{Z} + \mathcal{Z}^{\square} \star (\mathcal{B}_1 \star \mathcal{B}_1),$$

and verify that its EGF is $B_1(z) = \tan z$.

(b) Suppose instead that the left and right children are not distinguished. Show that the specification is

$$\mathcal{B}_2 \;=\; \mathcal{Z} + \mathcal{Z}^{\square} \star \mathrm{SET}_2(\mathcal{B}_2),$$

and verify that its EGF is $B_2(z) = \sqrt{2} \tan \frac{z}{\sqrt{2}}$.

**Example 18.** An *alternating permutation* (or *zig-zag permutation*) is a permutation $\sigma$ such that $\sigma_1 > \sigma_2 < \sigma_3 > \sigma_4 < \dots$. Let $a_n$ be the number of alternating permutations of length $n$. For instance, the alternating permutations of length 4 are $(2,1,4,3)$, $(3,1,4,2)$, $(3,2,4,1)$, $(4,1,3,2)$ and $(4,2,3,1)$, so $a_4 = 5$.

We will analyze alternating permutations using the min-boxed product. Let $\mathcal{A}_{\text{odd}}$ and $\mathcal{A}_{\text{even}}$ be the classes of alternating permutations of odd and even lengths, respectively.

In an alternating permutation with $n \geq 2$, the globally smallest element must occur at an even index (a "valley"). If we take an odd-length alternating permutation and cut it at its minimum element, the left and right parts both have odd lengths and are themselves alternating permutations. Thus,

$$\mathcal{A}_{\text{odd}} \;=\; \mathcal{Z} + \mathcal{Z}^{\square} \star (\mathcal{A}_{\text{odd}} \star \mathcal{A}_{\text{odd}}).$$

Using Theorem 13, we differentiate this to obtain the differential equation:

$$A'_{\text{odd}}(z) \;=\; 1 + A_{\text{odd}}(z)^2.$$

With the initial condition $A_{\text{odd}}(0) = 0$, the solution is exactly $A_{\text{odd}}(z) = \tan z$.

Similarly, if we cut an even-length alternating permutation at its minimum element, the left part is an odd-length alternating permutation, while the right part is an even-length alternating permutation.

This yields:

$$\mathcal{A}_{\text{even}} \;=\; \mathcal{E} + \mathcal{Z}^{\Box} \star (\mathcal{A}_{\text{odd}} \star \mathcal{A}_{\text{even}}).$$

Differentiating yields the linear differential equation:

$$A'_{\text{even}}(z) \;=\; A_{\text{odd}}(z)A_{\text{even}}(z) \;=\; \tan(z)A_{\text{even}}(z).$$

Separating variables gives $\frac{dA_{\text{even}}}{A_{\text{even}}} = \tan(z)dz$. Integrating yields $\ln A_{\text{even}}(z) = \ln(\sec z) + C$. Using the empty permutation base case $A_{\text{even}}(0) = 1$, we find $C = 0$, giving $A_{\text{even}}(z) = \sec z$.

Thus, the EGF for all alternating permutations, first discovered by Désiré André in 1881, is

$$A(z) \;=\; A_{\text{odd}}(z) + A_{\text{even}}(z) \;=\; \sec z + \tan z.$$

**Example 19.** Let $\mathcal{A}$ be the class of permutations with no decreasing run of length 3 in consecutive positions (i.e., no index $i$ such that $\sigma_i > \sigma_{i+1} > \sigma_{i+2}$). We can find its EGF using the min-boxed product.

If a permutation is non-empty, we cut the permutation at its globally smallest element 1, writing it as $\sigma = \pi 1 \tau$. Because 1 is the absolute minimum, the only way a decreasing run of length 3 could cross the cut at 1 is if $\pi$ ends in a descent. Thus, $\pi$ must belong to a more restrictive class $\mathcal{B}$: permutations in $\mathcal{A}$ that do *not* end in a descent. Since 1 is smaller than all elements in $\tau$, it forces an ascent, so $\tau$ can be any valid permutation in $\mathcal{A}$. This gives our first structural equation:

$$\mathcal{A} \;=\; \mathcal{E} + \mathcal{Z}^{\Box} \star (\mathcal{B} \star \mathcal{A}).$$

To find the structure of $\mathcal{B}$, we again decompose at the minimum element. A permutation in $\mathcal{B}$ is either empty ($\mathcal{E}$), or we write it as $\pi 1 \tau$. The prefix $\pi$ must still avoid ending in a descent (so $\pi \in \mathcal{B}$). The suffix $\tau$ must also avoid ending in a descent, but we must be careful: if $\tau$ is empty, then the whole permutation ends in 1. Since 1 is the minimum, the previous element in $\pi$ (if $\pi$ is non-empty) is larger, creating a descent at the end! To fix this, $\tau$ cannot be empty unless $\pi$ is also empty. If $\tau$ is empty and $\pi$ is non-empty, the invalid case is exactly $\mathcal{Z}^{\Box} \star (\mathcal{B} - \mathcal{E})$. Subtracting this invalid case from the decomposition $\mathcal{Z}^{\Box} \star (\mathcal{B} \star \mathcal{B})$ yields:

$$\mathcal{B} \;=\; \mathcal{E} + \mathcal{Z}^{\Box} \star (\mathcal{B} \star \mathcal{B} - \mathcal{B} + \mathcal{E}).$$

Using Theorem 13, these translate into a system of differential equations:

$$\begin{aligned} A'(z) &= B(z)A(z) \\ B'(z) &= B(z)^2 - B(z) + 1 \end{aligned}$$

with initial conditions $A(0) = 1$ and $B(0) = 1$. The second equation is a separable differential equation for $B(z)$:

$$\frac{\mathsf{d}B}{B^2 - B + 1} \;=\; dz \implies \int \frac{\mathsf{d}B}{(B - 1/2)^2 + 3/4} \;=\; z + C.$$

Integrating using arctangent and applying the initial condition yields:

$$\frac{2}{\sqrt{3}} \arctan\left(\frac{2B(z) - 1}{\sqrt{3}}\right) \;=\; z + \frac{\pi}{3\sqrt{3}}.$$

Solving for $B(z)$ and integrating $A'(z)/A(z) = B(z)$ produces the stunning closed-form EGF:

$$A(z) \;=\; \frac{\sqrt{3}}{2} \frac{e^{z/2}}{\cos\left(\frac{\sqrt{3}}{2}z + \frac{\pi}{6}\right)}.$$

**Exercise 20.** In the previous lecture, when we discussed Stirling numbers of the first kind, we showed that the bivariate EGF for permutations where $z$ marks the length and $u$ marks the number of records is $(1-z)^{-u}$. This was shown by constructing a bijection between permutations with a given number of records and permutations with the same number of cycles. Now use the min-boxed product decomposition to re-derive the bivariate EGF.

**Exercise 21.** Generalize the reasoning in Example 19 to find a system of differential equations for the class $\mathcal{A}$ of permutations with no decreasing run of length 4 in consecutive positions (i.e., no index $i$ such that $\sigma_i > \sigma_{i+1} > \sigma_{i+2} > \sigma_{i+3}$). Let $\mathcal{B}$ be the subclass of $\mathcal{A}$ consisting of permutations that do not end in a descent. Let $\mathcal{C}$ be the subclass of $\mathcal{A}$ consisting of permutations that do not end in a decreasing run of length 3 (so $\sigma_{n-2} > \sigma_{n-1} > \sigma_n$ is forbidden). Prove the following equalities, and translate them into a system of three coupled first-order differential equations for $A(z), B(z)$, and $C(z)$:

$$\begin{aligned}
\mathcal{A} &= \mathcal{E} + \mathcal{Z}^{\square} \star (\mathcal{C} \star \mathcal{A}) \\
\mathcal{B} &= \mathcal{E} + \mathcal{Z}^{\square} \star (\mathcal{C} \star \mathcal{B} - \mathcal{C} + \mathcal{E}) \\
\mathcal{C} &= \mathcal{E} + \mathcal{Z}^{\square} \star (\mathcal{C} \star \mathcal{C} - \mathcal{C} + \mathcal{B})
\end{aligned}$$

## Summary of Labelled Constructions

| **Construction** | **Combinatorial Class** | **Exponential Generating Function** |
|---|---|---|
| Disjoint Union | $\mathcal{A} + \mathcal{B}$ | $A(z) + B(z)$ |
| Labelled Product | $\mathcal{A} \star \mathcal{B}$ | $A(z)B(z)$ |
| Sequence | $\mathrm{SEQ}(\mathcal{A})$ | $\frac{1}{1-A(z)}$ |
| Set | $\mathrm{SET}(\mathcal{A})$ | $\exp(A(z))$ |
| Cycle | $\mathrm{CYC}(\mathcal{A})$ | $\log \frac{1}{1-A(z)}$ |
| Boxed Product | $\mathcal{A}^{\square} \star \mathcal{B}$ | $\int_0^z A'(t)B(t)\,\mathsf{d}t$ |