L6: Essential Complex Analysis for Analytic Combinatorics

Analytic Combinatorics
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Thus far, we have studied generating functions primarily as formal power series, extracting coefficients by algebraic methods such as recurrences, the binomial theorem and Lagrange inversion. These methods sometimes yield exact expressions, but many counting problems do not admit simple closed forms. In such cases, asymptotic estimates provide a natural description of the counting sequence: given $(a_n)_{n\geq 0}$, we seek a simple function $f(n)$ such that $\lim_{n\to\infty} a_n/f(n) = 1$.

Treating generating functions as *complex analytic functions* enables precise asymptotic estimates. We first explore the rigidity of complex differentiability and the relation between singularities and coefficient growth. We then develop contour integration and residues, which the next lecture will apply to generating functions with isolated poles. Some foundational results are stated without proof; see Bak and Newman [BN10] and Flajolet and Sedgewick [FS09, Chapter IV and Appendix B] for the omitted proofs and a more comprehensive treatment.

Complex Differentiability and Analyticity

For a real function $f : \mathbb{R} \to \mathbb{R}$, differentiability at x_0 is determined by the existence of the limit $\lim_{h\to 0}(f(x_0+h) - f(x_0))/h$, where h is real. For a complex function, the quotient has the same form, but h may approach 0 from any direction in the complex plane. The limit must therefore be independent of the path of approach. This additional requirement makes complex differentiability far more restrictive than real differentiability, and accounts for much of the rigidity of complex analytic functions.

Example 1. Consider $f_1(z) = z^2$ and $f_2(z) = \overline{z}^2$. When restricted to the real line, the two functions agree and are differentiable as real functions. In the complex plane, the corresponding difference quotients at z_0 are

$$\frac{f_1(z_0+h) - f_1(z_0)}{h} = 2z_0 + h, \qquad \frac{f_2(z_0+h) - f_2(z_0)}{h} = \frac{2\overline{z_0}h + \overline{h}^2}{h}.$$

The first quotient has limit $2z_0$ independently of how $h \to 0$. The second quotient has limit $2\overline{z_0}$ when h is real and $-2\overline{z_0}$ when h is purely imaginary. Thus, when $z_0 \neq 0$, the limit depends on the direction of approach.

In fact, f_1 is the unique function $f : \mathbb{C} \to \mathbb{C}$ that is complex differentiable everywhere and satisfies $f(x) = x^2$ for every $x \in \mathbb{R}$, by the identity theorem (Theorem 8) below. Agreement on any nontrivial real interval already suffices: *complex differentiability forces the function's values throughout the plane.* This illustrates the rigidity of complex analyticity.

We now formalize two a priori different notions of regularity for complex functions: *holomorphy*, expressed through complex differentiability, and *analyticity*, expressed through local power-series expansions.

Definition 2. Let $\Omega \subseteq \mathbb{C}$ be open. A function $f : \Omega \to \mathbb{C}$ is complex differentiable at $z_0 \in \Omega$ if

$$f'(z_0) = \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

exists for complex h. It is holomorphic on Ω if it is complex differentiable at every point of Ω.

Definition 3. Let $\Omega \subseteq \mathbb{C}$ be open. A function $f : \Omega \to \mathbb{C}$ is analytic at $z_0 \in \Omega$ if there exist a real number $R > 0$ and coefficients $(a_n)_{n\geq 0} \subseteq \mathbb{C}$, with $\{z : |z - z_0| < R\} \subseteq \Omega$, such that

$$f(z) \;=\; \sum_{n\geq 0} a_n (z - z_0)^n \qquad \text{for } |z - z_0| < R.$$

It is analytic on Ω if it is analytic at every point of Ω, and entire if it is analytic on all of $\mathbb{C}$.

Analyticity is useful because power series come with a stable calculus: inside their disk of convergence they define holomorphic functions, can be differentiated term by term and have uniquely determined coefficients. The next two standard results are proved in [BN10, Sections 2.3 and 6.1].

Theorem 4. Let $A(z) = \sum_{n\geq 0} a_n (z - z_0)^n$ have radius of convergence $R > 0$. Its sum is holomorphic for $|z - z_0| < R$, and it can be differentiated term by term:

$$A^{(k)}(z) \;=\; \sum_{n\geq k} \frac{n!}{(n-k)!} a_n (z - z_0)^{n-k}, \qquad k \geq 0.$$

The original series converges absolutely inside its convergence disk and uniformly on every smaller closed disk. Each differentiated series has radius of convergence R. In particular, $a_k = \frac{A^{(k)}(z_0)}{k!}$, so the coefficients of a power-series expansion are uniquely determined by the function.

Theorem 4 shows that analyticity implies holomorphy. The striking fact in complex analysis is the converse: complex differentiability alone forces a local power-series expansion.

Theorem 5. A function is analytic in an open set $\Omega \subseteq \mathbb{C}$ if and only if it is holomorphic in Ω. More precisely, if f is holomorphic throughout a disk $|z - z_0| < r$, then its Taylor series at z_0 represents f throughout that disk.

Theorem 5 has several important consequences:

- One complex derivative implies derivatives of all orders: A function that is complex differentiable throughout Ω is automatically infinitely differentiable there. This implication fails for real functions.[1]
- A local power series determines the function on its disk of convergence: In real analysis, the derivatives at a point need not determine a function away from that point.[2] For a holomorphic function f, the local power series $\sum_{n\geq 0} a_n (z - z_0)^n$ defines a holomorphic extension of the local function throughout its disk of convergence.
- Identity theorem and rigidity: The identity theorem (Theorem 8 below) strengthens this local picture: a holomorphic extension to a specified connected domain is uniquely determined, if it exists. Thus, once a generating function is known near the origin, its values on any such domain are fixed. The next section develops this principle through analytic continuation. Later, we will examine the points where continuation fails, because their location and local behaviour govern the asymptotic growth of the coefficients.

[1] A classic counterexample is $f(x) = x|x|$.

[2] For example, for $\epsilon > 0$, define $f_\epsilon(x) = x^2$ if $|x| \leq \epsilon$, $f_\epsilon(x) = 2\epsilon x - \epsilon^2$ if $x > \epsilon$ and $f_\epsilon(x) = -2\epsilon x - \epsilon^2$ if $x < -\epsilon$. These functions are differentiable throughout $\mathbb{R}$, and they all have the same Taylor series around 0. However, for any fixed $x \neq 0$, the value $f_\epsilon(x)$ depends on ϵ once $\epsilon < |x|$.

Identity Theorem and Analytic Continuation

The equivalence of holomorphy and analyticity gives a local power-series representation of every holomorphic function. We now ask how far the function represented by such a series can be extended beyond its disk of convergence. Analytic continuation provides the framework for such extensions, while the identity theorem ensures that an extension to a given connected domain is unique.

Example 6. The power series $G(z) = \sum_{n\geq 0} z^n$ defines a holomorphic function only on $|z| < 1$. On that disk it equals $1/(1-z)$, which is holomorphic on the larger connected domain $\mathbb{C} \setminus \{1\}$. The rational expression extends the function even though the original Taylor series retains radius 1.

Example 7. Analytic continuation is not limited to functions with simple rational closed forms. Recall the *tree function*

$$T(z) \;=\; \sum_{n\geq 1} n^{n-1} \frac{z^n}{n!},$$

which counts rooted labelled trees. This power series is characterized implicitly by $T(z) = ze^{T(z)}$. Its Taylor series has radius $1/e$, but the function it defines near 0 admits analytic continuation to the slit plane $\mathbb{C} \setminus [1/e, \infty)$ [Natb]. We will not prove this fact here; the point is that the implicit equation controls the continuation even though no elementary closed form is available. Thus analytic continuation can reveal a much larger domain than the original Taylor disk.

These examples show that a function may extend beyond the convergence disk of its Taylor series. The key structural issue is uniqueness: if two holomorphic functions on the same connected domain extend the same local function, the identity theorem forces them to agree throughout that domain.

Theorem 8. Identity Theorem [BN10, Section 6.3]. Let Ω be a connected open set, and let f and g be holomorphic on Ω. If $f = g$ on a set having an accumulation point in Ω, then $f = g$ throughout Ω.

Thus an extension to a specified connected domain, if it exists, is uniquely determined by the original function.

Definition 9. Let f be holomorphic on an open connected set Ω. If $\widetilde{\Omega}$ is a domain (a connected open set) with $\Omega \subseteq \widetilde{\Omega}$ and a holomorphic function F on $\widetilde{\Omega}$ satisfies $F(z) = f(z)$ for every $z \in \Omega$, then F is an analytic continuation of f to $\widetilde{\Omega}$. By Theorem 8, such a continuation is unique on the specified domain $\widetilde{\Omega}$.

Singularities, Convergence Radii and Coefficient Growth

Analytic continuation can extend a function beyond its original convergence disk, but it may encounter obstructions. We now relate these obstructions to the radius of convergence and, through that radius, to the growth of the coefficients. For now, we use singularity informally for a boundary point across which the given function cannot be extended holomorphically. Later in this lecture, we will distinguish poles from other kinds of isolated singularities. The proofs of Proposition 10 and Theorems 12 and 13 are deferred to an appendix at the end of this note.

Proposition 10. If $A(z) = \sum_{n\geq 0} a_n z^n$ has finite positive radius of convergence R, then at least one point on $|z| = R$ is a singularity.

Example 11. Both $1/(1-z) = \sum_{n\geq 0} z^n$ and $1/(1+z) = \sum_{n\geq 0}(-1)^n z^n$ have radius 1. The first is singular at 1, whereas the second is holomorphic there and singular at -1. Nonnegative coefficients force a singularity on the positive real axis, as the Vivanti–Pringsheim Theorem below shows.

Theorem 12. Vivanti–Pringsheim Theorem. Let $A(z) = \sum_{n\geq 0} a_n z^n$ be a power series with coefficients $a_n \geq 0$ and finite positive radius of convergence R. Then $z = R$ is a singularity of A: there is no disk D centered at R and holomorphic function on D agreeing with A on $D \cap \{z : |z| < R\}$.

The sign condition is essential: in Example 11, alternating signs allow the second function to remain regular at the positive boundary point. When a counting generating function has finite positive radius R, this number is also an analytic obstruction on the positive real axis. There may be other singularities on the same circle, and in extreme cases the entire boundary can be obstructed; see Example 16 below. The same number R also determines the upper exponential growth rate of the coefficients.

Theorem 13. Cauchy–Hadamard Formula. Let $A(z) = \sum_{n\geq 0} a_n z^n$ have radius of convergence R, with the conventions $1/0 = \infty$ and $1/\infty = 0$. Then

$$\frac{1}{R} \;=\; \limsup_{n\to\infty} |a_n|^{1/n}.$$

Consequently, the reciprocal radius R^{-1} is the limsup exponential growth rate of the coefficients. The following example illustrates why the limsup is needed.

Example 14. For $A(z) = 1/(1-2z)$, the coefficients are $a_n = 2^n$ and the radius is $R = 1/2$, so $a_n^{1/n} = 1/R$. For $B(z) = 1/(1-4z^2)$, the radius is also $1/2$, but the odd coefficients vanish while $[z^{2m}]B(z) = 4^m$. Its coefficient roots therefore alternate between 0 and 2: they do not converge, but their limsup is $2 = 1/R$, as predicted by Theorem 13.

Exercise 15. Determine the radii of convergence of

$$F(z) \;=\; \sum_{n\geq 1} \frac{z^n}{n^2}, \qquad G(z) \;=\; \sum_{n\geq 0} \frac{z^n}{n!}, \qquad H(z) \;=\; \sum_{n\geq 0} n! z^n.$$

Use Theorem 13 and, where applicable, Theorem 12. Explain why $F(1) < \infty$ is consistent with 1 being a singularity of F. Finally, compare $1/(1-z)$ and $1/(1-z)^2$: what information about their coefficients does their common radius capture, and what does it miss?

Here is an amusing fact without proof. The function $F = \mathrm{Li}_2$ is the *dilogarithm*, which admits analytic continuation to $\mathbb{C} \setminus [1, +\infty)$ [Nata]. For fixed $\epsilon > 0$, as $r \to +\infty$, we have $\mathrm{Li}_2(r + i\epsilon) = -\frac{1}{2}(\log r)^2 + \frac{\pi^2}{3} + i\pi \log r + o(1)$. Its real part thus tends to $-\infty$, despite its positive Taylor coefficients!

Example 16. The series

$$L(z) \;=\; \sum_{k\geq 0} z^{2^k}$$

satisfies $L(z) = z + L(z^2)$ on $|z| < 1$. Its coefficients are 0 or 1, with infinitely many equal to 1, so Theorem 13 gives radius of convergence 1. The unit circle is a natural boundary for L, meaning that L admits no analytic continuation across any point of that circle.

To see this, first observe that $L(r) \to \infty$ as $r \uparrow 1$, because any fixed number of its nonnegative summands tend to 1. If $\zeta^{2^m} = 1$ for an integer $m \geq 1$, then for $0 < r < 1$,

$$L(r\zeta) \;=\; \sum_{k=0}^{m-1} (r\zeta)^{2^k} + L(r^{2^m}).$$

The finite sum remains bounded, whereas the last term tends to infinity. Thus $|L(r\zeta)| \to \infty$, so every 2^m-th root of unity is a singularity. These roots are dense on the unit circle. Any holomorphic extension across a boundary point would therefore extend L across one of these roots, a contradiction.

This phenomenon also occurs in combinatorics. Euler's partition OGF $\prod_{j\geq 1}(1 - z^j)^{-1}$ has radius 1,[3] and a singularity at every root of unity, so the unit circle is again a natural boundary. We omit the estimates needed for this assertion [FS09, Note IV.17 and Section VIII.6].

Meromorphic Functions and Laurent Series

The radius of convergence determines an exponential scale, but it does not determine leading constants or polynomial factors. For that finer information, we must examine the local form of each singularity. We begin with poles, which are the simplest obstructions to holomorphy and occur naturally in quotients of analytic functions.

Example 17. The function $1/(1 - z)$ is not holomorphic at $z = 1$, but multiplying by $z - 1$ gives the constant -1. Similarly, $(1 - z)^{-3}$ becomes holomorphic after multiplication by $(z - 1)^3$, but not by a lower positive power. The necessary power measures the order of the pole.

For a combinatorial example, the derangement EGF $D(z) = e^{-z}/(1 - z)$ has the same type of singularity at 1 as the geometric series: $(z - 1)D(z) = -e^{-z}$ is holomorphic and nonzero there. Multiplying by an analytic numerator changes the local constant without changing the pole order, provided the numerator does not vanish.

Definition 18. A function f is meromorphic on an open set Ω if there is a discrete set $P \subseteq \Omega$, meaning a set with no accumulation point in Ω, such that f is holomorphic on $\Omega \setminus P$ and, for each $z_0 \in P$, there are an integer $m \geq 1$ and a function g holomorphic near z_0 with $g(z_0) \neq 0$ such that

$$f(z) \;=\; \frac{g(z)}{(z - z_0)^m} \qquad \text{for } 0 \;<\; |z - z_0| \text{ sufficiently small.}$$

The set P may be infinite, but its points cannot accumulate inside Ω. The point z_0 is then a pole of order m. A pole of order 1 is called a simple pole.

[3] An elementary but nontrivial combinatorics exercise: show that the partition numbers $p(n)$ satisfy $1 \leq p(n) \leq (1 + \epsilon)^n$ for each $\epsilon > 0$ and all sufficiently large n. Hence $p(n)^{1/n} \to 1$, and the radius is exactly 1 by Theorem 13.

Meromorphic functions need not be rational. The exponential function is entire, so quotients such as $1/(2-e^z)$ are meromorphic wherever their denominator zeros are isolated.

Poles are only one kind of isolated singularity. To see what distinguishes them, compare the negative powers in the following local expansions.

Example 19. On the punctured disk $0 < |z| < 1$,

$$\frac{1}{z^2(1-z)} = z^{-2} + z^{-1} + 1 + z + \cdots, \qquad e^{1/z} = 1 + z^{-1} + \frac{z^{-2}}{2!} + \cdots.$$

Multiplying the first function by z^2 removes its singularity. No finite power of z can remove all the negative powers in the second expansion. In contrast, $\sin z/z = 1 - z^2/3! + \cdots$ has no negative powers at all: its apparent singularity at 0 disappears when we assign it the value 1 there.

Definition 20. A point z_0 is an isolated singularity of f if f is holomorphic on some punctured disk $0 < |z - z_0| < R$ but is not holomorphic at z_0. The singularity is removable if f admits a holomorphic extension across z_0. An isolated singularity that is neither removable nor a pole is called essential.

Taylor series describe holomorphic functions inside disks. Laurent series are the corresponding expansions on annuli, where negative powers record the behaviour towards the missing center. They can often be computed by rewriting a function using familiar geometric or exponential series, as in Example 19. The following standard theorem justifies this approach [BN10, Chapter 9].

Theorem 21. A function holomorphic on an annulus $r < |z - z_0| < R$, where $0 \le r < R$, has a unique Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n,$$

converging absolutely there and uniformly on compact subsets. The sum of its negative-power terms is the principal part.

For the expansion on a punctured disk ($r = 0$), an isolated singularity at z_0 is removable exactly when all negative coefficients vanish, a pole of order m exactly when $c_{-m} \neq 0$ and $c_n = 0$ for $n < -m$, and essential exactly when infinitely many negative coefficients are nonzero.

Unlike the formal Laurent series used in Lecture 5, an analytic Laurent expansion may have infinitely many negative powers. A pole is precisely the case with a finite nonzero principal part. This is why pole calculations reduce to finitely many local coefficients; essential singularities require other methods.

We conclude with an example showing how a functional equation constructs an analytic continuation.

Example 22. For $\Re z > 0$, the Gamma function is defined by

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} \, \mathrm{d}t.$$

This integral defines a holomorphic function on $\Re z > 0$ [BN10, Section 18.3]. We now use its recurrence to construct an analytic continuation.

Integration by parts gives $\Gamma(z+1) = z\Gamma(z)$ for $\Re z > 0$. For any integer $N \geq 1$, iterating this identity gives

$$\Gamma(z) \;=\; \frac{\Gamma(z+N)}{z(z+1)\cdots(z+N-1)}$$

initially for $\Re z > 0$. The right-hand side defines an extension wherever $\Re(z+N) > 0$ and the denominator is nonzero. The recurrence shows that these formulas agree for consecutive values of N on their common domains; hence they agree for all N on overlaps. Their domains cover the plane except for the nonpositive integers. Near a nonpositive integer $-k$, choose $N > k$. The denominator contains exactly one vanishing factor, $z+k$, while the numerator is holomorphic and has value $\Gamma(N-k) = (N-k-1)! > 0$ at $z = -k$, by the recurrence and $\Gamma(1) = 1$. Thus Γ extends meromorphically to $\mathbb{C}$, with a simple pole at each of $0, -1, -2, \ldots$.

Since $\Gamma(1/2) = \sqrt{\pi}$, as follows from the Gaussian integral, the recurrence gives

$$\Gamma\left(-\frac{1}{2}\right) \;=\; -2\Gamma\left(\frac{1}{2}\right) \;=\; -2\sqrt{\pi}.$$

Cauchy's Theorems and Coefficient Integrals

We have related singularities to coefficient growth and described their local expansions. To turn this information into precise coefficient estimates, we need an expression for a coefficient in terms of function values. The starting point is the familiar orthogonality of the complex exponentials on a circle: averaging around the circle isolates a single coefficient.

Example 23. Let $A(z) = \sum_{m\geq 0} a_m z^m$ be convergent for $|z| < R$, and fix $0 < r < R$. Then, for every $n \geq 0$,

$$\frac{1}{2\pi}\int_0^{2\pi} A(re^{it})e^{-int}\,\mathsf{d}t \;=\; \sum_{m\geq 0} a_m r^m \frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)t}\,\mathsf{d}t \;=\; a_n r^n.$$

The interchange of summation and integration is justified because $r < R$, so the power series converges absolutely and uniformly on the circle $|z| = r$. Thus averaging around a circle isolates exactly one coefficient. Contour integration is the coordinate-free way to express this circular averaging.

Complex integration is ordinary real integration after a curve has been parametrized. The factor $\gamma'(t)$ records both the direction and the local displacement along the curve.

Definition 24. A contour is a piecewise continuously differentiable map $\gamma : [a,b] \to \mathbb{C}$. It is closed if $\gamma(a) = \gamma(b)$. If f is continuous on the image of γ, its contour integral along γ is

$$\int_\gamma f(z)\,\mathsf{d}z \;=\; \int_a^b f(\gamma(t))\gamma'(t)\,\mathsf{d}t.$$

For a closed contour, we write $\oint_\gamma f(z)\,\mathsf{d}z$. A closed contour is simple if it has no self-intersections other than its coinciding endpoints. A simple closed contour is positively oriented when it is traversed counterclockwise.

The following elementary estimate is the basic tool for proving that remote parts of a contour contribute little to coefficient asymptotics.

Proposition 25. ML Inequality. Let γ be a contour of length L, and let f be continuous on γ. If $|f(z)| \le M$ for every point z on γ, then

$$\left| \int_\gamma f(z)\,\mathsf{d}z \right| \;\le\; ML.$$

Proof: If $\gamma : [a,b] \to \mathbb{C}$ parametrizes the contour, then

$$\left| \int_\gamma f(z)\,\mathsf{d}z \right| \;=\; \left| \int_a^b f(\gamma(t))\gamma'(t)\,\mathsf{d}t \right| \;\le\; \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,\mathsf{d}t \;\le\; M \int_a^b |\gamma'(t)|\,\mathsf{d}t \;=\; ML. \qquad \square$$

Example 26. A direct calculation shows why the power z^{-1} is special. If $\gamma(t) = e^{it}$ for $0 \le t \le 2\pi$, then for any integer k,

$$\oint_\gamma z^k\,\mathsf{d}z \;=\; i\int_0^{2\pi} e^{i(k+1)t}\,\mathsf{d}t \;=\; \begin{cases} 2\pi i, & k = -1, \\ 0, & k \ne -1. \end{cases}$$

For negative k, the function z^k is not holomorphic at 0; this is why a nonzero integral is possible. Integration around the circle detects precisely the coefficient of z^{-1}.

One of the most beautiful features of complex analysis is that *a closed contour can be stretched and bent without changing the integral*, provided the deformation stays within a region where the integrand is holomorphic. The curve may look entirely different, yet the integral remains exactly the same. Cauchy's theorems make this remarkable geometric freedom precise. We use the following standard forms [BN10, Chapter 8].

Theorem 27. Cauchy's Integral Theorem. Let $\Omega \subseteq \mathbb{C}$ be open and simply connected, meaning informally that it has no holes, and let f be holomorphic on Ω. For every closed contour γ in Ω,

$$\oint_\gamma f(z)\,\mathsf{d}z \;=\; 0.$$

Cauchy's integral formula recovers a holomorphic function from its values on a surrounding contour.

Theorem 28. Cauchy's Integral Formula [BN10, Sections 6.1 and 10.2]. Let γ be a positively oriented simple closed contour whose image and interior lie in an open set Ω, and let f be holomorphic on Ω. If z_0 lies inside γ, then for all $n \ge 0$,

$$f^{(n)}(z_0) \;=\; \frac{n!}{2\pi i} \oint_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\,\mathsf{d}\zeta.$$

Cauchy's integral formula is stronger than it first appears: values on a surrounding contour determine every derivative at an interior point. It also explains Theorem 5: on a circle centered at z_0, expand $(\zeta - z)^{-1}$ as a geometric series in $(z - z_0)/(\zeta - z_0)$ and integrate term by term. Uniform convergence on the circle justifies the interchange, and the derivative formula identifies the coefficients as $f^{(n)}(z_0)/n!$.

For analytic combinatorics, the most important special case of Cauchy's formula is obtained by placing the center at the origin. It turns coefficient extraction into a contour integral.

Corollary 29. Cauchy's Coefficient Formula. If $A(z) = \sum_{n\geq 0} a_n z^n$ is holomorphic on an open set containing a positively oriented simple closed contour γ and its interior, and if the origin lies inside γ, then for all $n \geq 0$,

$$a_n \;=\; [z^n]A(z) \;=\; \frac{1}{2\pi i}\oint_\gamma \frac{A(z)}{z^{n+1}}\,\mathsf{d}z.$$

This formula is the bridge from analysis back to enumeration. Once a_n is written as a contour integral, bounds on $A(z)$ along a circle immediately become bounds on a_n. The hypotheses ensure that $A(z)/z^{n+1}$ has no singularities on or inside the contour except a possible pole at the origin; this singularity is removable if A has a zero of order at least $n+1$ there.

Combining Cauchy's coefficient formula with the ML inequality immediately gives coefficient bounds.

Exercise 30. Let $r > 0$, and suppose $A(z) = \sum_{n\geq 0} a_n z^n$ is holomorphic on a neighborhood of the closed disk $|z| \leq r$. Let $M_r = \max_{|z|=r} |A(z)|$. Use Corollary 29 and Proposition 25 to prove the *Cauchy estimate*

$$|a_n| \;\leq\; \frac{M_r}{r^n}.$$

Residues and the Contributions of Singularities

Cauchy's coefficient formula applies when the generating function is holomorphic inside the contour. In applications, however, we want to enlarge that contour beyond its original convergence disk. Crossing an isolated singularity changes the integral by a quantity determined by one Laurent coefficient. This provides the connection between local singular behaviour and global coefficient estimates.

On a small circle around z_0, a Laurent expansion converges uniformly, so it can be integrated term by term. The calculation in Example 26, after translating the center, shows that every power of $z - z_0$ integrates to zero except $(z - z_0)^{-1}$. Thus the entire principal part contributes through a single coefficient.

Definition 31. For a function holomorphic on a punctured disk about z_0, its residue is the coefficient

$$\operatorname{Res}(f; z_0) \;=\; c_{-1}$$

in its Laurent expansion there. At a regular point or removable singularity, this coefficient is zero.

Example 32. In Example 19, both $1/(z^2(1-z))$ and $e^{1/z}$ have residue 1 at 0, despite having different types of singularity. The residue of $\sin z/z$ at 0 is 0. A residue measures the contribution to a contour integral, rather than the severity of the singularity.

Cauchy's theorem says that a holomorphic function contributes nothing around a closed contour enclosing no holes in its domain. The residue theorem gives the correction when isolated singularities are present: each contributes its own local coefficient. We state this theorem without proof.

Theorem 33. Residue Theorem [BN10, Chapter 10]. Let γ be a positively oriented simple closed contour. Suppose f is holomorphic on an open neighborhood of γ and its interior except at finitely many isolated singularities $z_1, \ldots, z_k$ strictly inside γ. Then

$$\oint_\gamma f(z)\,\mathsf{d}z \;=\; 2\pi i \sum_{j=1}^{k} \operatorname{Res}(f; z_j).$$

Under the hypotheses of Theorem 33, positively oriented contours enclosing the same singularities give the same integral. This explains contour deformation: a contour can be moved through a region where the integrand is holomorphic without changing the integral. Enlarging the contour to enclose additional isolated singularities adds $2\pi i$ times the sum of their residues.

For poles, the residue can be computed from ordinary derivatives. For instance, in Example 19, the residue of $1/(z^2(1-z))$ at 0 is the coefficient of z in $1/(1-z)$, namely 1.

Proposition 34. If f has a pole of order m at z_0, then

$$\operatorname{Res}(f; z_0) \;=\; \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{\mathsf{d}^{m-1}}{\mathsf{d}z^{m-1}} \big((z-z_0)^m f(z)\big).$$

If $f = g/h$ near z_0, with g, h holomorphic, $h(z_0) = 0$ and $h'(z_0) \neq 0$, then

$$\operatorname{Res}(f; z_0) \;=\; \frac{g(z_0)}{h'(z_0)}.$$

Proof: Writing $f(z) = g(z)/(z-z_0)^m$, the residue is the coefficient of $(z-z_0)^{m-1}$ in g, namely $g^{(m-1)}(z_0)/(m-1)!$. For the second formula, factor $h(z) = (z-z_0)\widetilde{h}(z)$ with $\widetilde{h}(z_0) = h'(z_0) \neq 0$; the residue is the value of $g/\widetilde{h}$ at z_0. □

The radius of convergence supplies an exponential scale; contour integrals and residues recover the finer information. In the next lecture, we will use these tools to obtain exact formulas for rational generating functions and asymptotic estimates for meromorphic ones, keeping track of how the location and order of each pole affect the answer.

References

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Appendix: Deferred Proofs

Proof of Proposition 10: Suppose that every boundary point admits a holomorphic continuation to a disk centered there. Any two overlapping such disks have a connected intersection meeting $|z| < R$, so their continuations agree by Theorem 8. They therefore combine with A into one holomorphic function on an open neighborhood of the closed disk $|z| \le R$. By compactness, this neighborhood contains $|z| < R + \epsilon$ for some $\epsilon > 0$. By Theorem 5, the Taylor series at 0 converges throughout this larger disk. Its coefficients are the original a_n, contradicting the radius R. □

Proof of Theorem 12: Suppose instead that A extends holomorphically across R. By Theorem 5, for some $\epsilon > 0$, it has a local Taylor expansion for all $|t| < \epsilon$:

$$A(R+t) \;=\; \sum_{k\ge 0} b_k t^k, \qquad \text{where } b_k \;=\; \frac{A^{(k)}(R)}{k!}.$$

For $0 < r < R$, termwise differentiation gives

$$\frac{A^{(k)}(r)}{k!} \;=\; \sum_{n\ge k} \binom{n}{k} a_n r^{n-k}.$$

As $r \uparrow R$, continuity of the derivatives gives the limit b_k on the left. On the right, all summands are nonnegative and nondecreasing in r, so monotone convergence yields

$$b_k \;=\; \sum_{n\ge k} \binom{n}{k} a_n R^{n-k} \;\ge\; 0.$$

Fix a real t with $0 < t < \epsilon$. The Taylor series at R converges at $R + t$. Rearranging its nonnegative double sum is valid because either order of summation is the supremum of all finite subsums. Hence the binomial theorem gives

$$\infty \;>\; \sum_{k\ge 0} b_k t^k \;=\; \sum_{n\ge 0} a_n \sum_{k=0}^{n} \binom{n}{k} R^{n-k} t^k \;=\; \sum_{n\ge 0} a_n (R+t)^n.$$

For every $z \in \mathbb{C}$ with $|z| < R + t$, nonnegativity gives $|a_n z^n| = a_n |z|^n \le a_n (R+t)^n$ for all n. Comparison with the convergent series above therefore proves absolute convergence throughout the disk $|z| < R + t$. The radius of convergence is thus at least $R + t > R$, contradicting the assumed radius R. □

Proof of Theorem 13: Write $L = \limsup_{n\to\infty} |a_n|^{1/n} \in [0, \infty]$ and fix $z \neq 0$. If $L < 1/|z|$, choose q with $L < q < 1/|z|$. By the definition of the limit superior, $|a_n|^{1/n} \le q$ for all sufficiently large n. Consequently

$$|a_n z^n| \;\le\; (q|z|)^n, \qquad q|z| \;<\; 1,$$

so comparison with a geometric series proves absolute convergence. If $L > 1/|z|$, infinitely many n satisfy $|a_n|^{1/n} > 1/|z|$, so $|a_n z^n| > 1$. The terms do not tend to zero, so the series diverges. For $0 < L < \infty$, convergence for $|z| < 1/L$ and divergence for $|z| > 1/L$ give $R = 1/L$. If $L = 0$, the series converges for every z, so $R = \infty$. If $L = \infty$, it diverges for every $z \neq 0$, so $R = 0$. □