L7: Coefficient Asymptotics from Isolated Poles

Analytic Combinatorics
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We now apply the tools of complex analysis developed in Lecture 6 to estimate the coefficients of generating functions asymptotically. We begin with functions whose singularities are isolated poles, with rational functions as an important special case.

Coefficient Formulas and Pole Contributions

We recall three results from Lecture 6. Together, they turn coefficient extraction into a sum of pole contributions with a controlled error:

- **Cauchy's coefficient formula.** If A is holomorphic on a neighborhood of $|z| \le r$, where $r > 0$, then for every $n \ge 0$,
$$[z^n]A(z) \;=\; \frac{1}{2\pi i}\oint_{|z|=r} \frac{A(z)}{z^{n+1}}\,\mathsf{d}z.$$
- **Residue theorem.** Let γ be a positively oriented simple closed contour. If f is holomorphic on a neighborhood of γ and its interior except at finitely many isolated singularities $z_1, \ldots, z_k$ strictly inside γ, then
$$\oint_\gamma f(z)\,\mathsf{d}z \;=\; 2\pi i \sum_{j=1}^{k} \operatorname{Res}(f; z_j).$$
- **ML inequality.** If f is continuous on a contour γ of length L and $|f(z)| \le M$ there, then
$$\left|\int_\gamma f(z)\,\mathsf{d}z\right| \;\le\; ML.$$

Choose $r > 0$ smaller than the radius of convergence of A. Suppose A extends meromorphically to a neighborhood of $|z| \le R$, where $R > r$, with no poles on $|z| = R$. Beyond the convergence disk, we use the analytic continuation of A. Denote the enclosed poles by $\rho_1, \ldots, \rho_k$. With both circles oriented counterclockwise, Cauchy's coefficient formula and the residue theorem give

$$[z^n]A(z) \;=\; \frac{1}{2\pi i}\oint_{|z|=r} \frac{A(z)}{z^{n+1}}\,\mathsf{d}z \;=\; \frac{1}{2\pi i}\oint_{|z|=R} \frac{A(z)}{z^{n+1}}\,\mathsf{d}z - \sum_{j=1}^{k} \operatorname{Res}\left(\frac{A(z)}{z^{n+1}}; \rho_j\right).$$

Since A has no pole on $|z| = R$, the maximum $M_R = \max_{|z|=R} |A(z)|$ is finite. The ML inequality gives

$$\left|\frac{1}{2\pi i}\oint_{|z|=R} \frac{A(z)}{z^{n+1}}\,\mathsf{d}z\right| \;\le\; \frac{1}{2\pi}\cdot 2\pi R \cdot \frac{M_R}{R^{n+1}} \;=\; M_R R^{-n}.$$

For fixed R, this is precisely the error estimate in the following theorem.

Theorem 1. Let A be holomorphic at 0 and meromorphic on a neighborhood of the closed disk $|z| \le R$, where $R > 0$, with no pole on $|z| = R$. Let $\rho_1, \ldots, \rho_k$ be its poles inside the disk. Then, as $n \to \infty$ with R fixed,
$$[z^n]A(z) \;=\; -\sum_{j=1}^{k} \operatorname{Res}\left(\frac{A(z)}{z^{n+1}}; \rho_j\right) + \mathcal{O}(R^{-n}).$$

For rational functions (such as the OGFs of regular languages in Lecture 1), partial fractions give an *exact* algebraic version of this result. The principal part at a pole ρ consists of the negative powers of $z-\rho$ in its Laurent expansion. We write it using $(1-z/\rho)^{-j} = (-\rho)^j (z-\rho)^{-j}$, with constants adjusted accordingly, to make coefficient extraction immediate.

Theorem 2. Let $A(z) = U(z)/V(z)$, where U and V are polynomials with no common root and $V(0) \neq 0$. Thus A is holomorphic at 0. Its partial-fraction decomposition has the form

$$A(z) \;=\; Q(z) + \sum_{\rho} \sum_{j=1}^{m_\rho} \frac{c_{\rho,j}}{(1-z/\rho)^j},$$

where Q is a polynomial, ρ ranges over the roots of V and m_ρ is the multiplicity of that root. Since U and V have no common root, each ρ is a pole of order m_ρ. Then

$$[z^n]A(z) \;=\; [z^n]Q(z) + \sum_{\rho} \sum_{j=1}^{m_\rho} c_{\rho,j} \binom{n+j-1}{j-1} \rho^{-n}.$$

Thus, apart from finitely many initial coefficients contributed by Q, the coefficient sequence is a sum of terms $P_\rho(n)\rho^{-n}$, where P_ρ has degree $m_\rho - 1$. If $\deg U < \deg V$, then $Q = 0$ and the pole terms give every coefficient exactly.

Proof: Dividing U by V first separates the polynomial part Q, and partial fractions separate the poles. Differentiating the geometric series $j-1$ times gives

$$[z^n](1-z/\rho)^{-j} \;=\; \binom{n+j-1}{j-1} \rho^{-n} \qquad (n \geq 0,\ j \geq 1).$$

Applying this identity to each partial-fraction term proves the coefficient formula. The binomial coefficient is a polynomial in n of degree $j-1$. The highest-order principal-part coefficient c_{ρ,m_ρ} is nonzero because the pole order is m_ρ. Thus P_ρ has degree exactly $m_\rho - 1$, with leading coefficient $c_{\rho,m_\rho}/(m_\rho-1)!$. □

Dominant Poles and Error Estimates

The two coefficient formulas separate the effects of pole location and pole order: location controls the exponential scale, while order controls the degree of the polynomial factor. We now collect the contributions from all poles closest to the origin.

Definition 3. For a function holomorphic at 0 whose nearest singularities are poles, those of smallest modulus are its dominant poles.

Writing $\rho = re^{i\theta}$ gives $\rho^{-n} = r^{-n}e^{-in\theta}$. Poles of the same modulus therefore contribute at the same exponential scale, but with different phases. The phase factors are periodic when $e^{i\theta}$ is a root of unity. In general they may oscillate without being periodic.

For a power series with nonnegative coefficients and a finite positive radius of convergence, the Vivanti–Pringsheim theorem from Lecture 6 ensures a singularity at the positive real point equal to that radius. If the function is meromorphic on a neighborhood of the closed convergence disk, this singularity is a pole. The theorem does not exclude other poles on the same circle. The following result retains *all* dominant poles and controls the error using the distance to the next poles.

Corollary 4. Let A be holomorphic at 0 and meromorphic on a disk $|z| < S$, where S may be infinite. Suppose its dominant poles are $\rho_1, \ldots, \rho_k$, all of modulus r_1. Let r_2 be the smallest modulus of any other pole in this disk, or set $r_2 = S$ if there are no others. If ρ_ℓ has order m_ℓ, write its principal part as

$$\sum_{j=1}^{m_\ell} \frac{c_{\ell,j}}{(1 - z/\rho_\ell)^j}.$$

For any fixed $r_1 < R < r_2$, as $n \to \infty$,

$$[z^n]A(z) \;=\; \sum_{\ell=1}^{k} \sum_{j=1}^{m_\ell} c_{\ell,j} \binom{n+j-1}{j-1} \rho_\ell^{-n} + \mathcal{O}(R^{-n}).$$

In particular, let $m = \max_{1 \le \ell \le k} m_\ell$. The first-order form is

$$[z^n]A(z) \;=\; \frac{n^{m-1}}{(m-1)!} \sum_{\ell:\, m_\ell = m} c_{\ell,m} \rho_\ell^{-n} + o\big(n^{m-1} r_1^{-n}\big)\,.$$

If all dominant poles are simple, then

$$[z^n]A(z) \;=\; -\sum_{\ell=1}^{k} \operatorname{Res}(A; \rho_\ell) \rho_\ell^{-n-1} + \mathcal{O}(R^{-n}).$$

Proof: Subtract all dominant principal parts, defining

$$H(z) \;=\; A(z) - \sum_{\ell=1}^{k} \sum_{j=1}^{m_\ell} \frac{c_{\ell,j}}{(1 - z/\rho_\ell)^j}.$$

The singularities at the ρ_ℓ are removable, and there are no other poles in $|z| < r_2$. After filling in the removable singularities, H is holomorphic on a neighborhood of $|z| \le R$. The Cauchy estimate gives $|[z^n]H(z)| \le \max_{|z|=R} |H(z)| R^{-n} = \mathcal{O}(R^{-n})$. Extracting the coefficients of the principal parts as in Theorem 2 proves the first formula. For a simple pole, $c_{\ell,1} = -\operatorname{Res}(A; \rho_\ell)/\rho_\ell$.

For the first-order form, $\binom{n+j-1}{j-1}$ is a polynomial of degree $j-1$ with leading coefficient $1/(j-1)!$. Thus only terms with $j = m$ contribute at order $n^{m-1} r_1^{-n}$; all lower-degree terms and the exponentially smaller error $\mathcal{O}(R^{-n})$ are $o(n^{m-1} r_1^{-n})$. □

Alignments and Permutations

We now apply these results to generating functions from earlier lectures. In each case, we locate the dominant poles, compute their residues and bound the error from the remaining analytic part.

Example 5. An *alignment* is an ordered partition of a labelled set into nonempty blocks. From the sequence construction in Lecture 3, its EGF is the meromorphic function already mentioned in Lecture 6:

$$A(z) \;=\; \frac{1}{2 - e^z}, \qquad a_n \;=\; n![z^n]A(z).$$

The poles occur where $e^z = 2$, namely at $\log 2 + 2\pi i k$ for $k \in \mathbb{Z}$. Each is simple because the derivative of the denominator is $-e^z = -2$ there. The nearest pole is the positive real point $\alpha = \log 2$; the next

pair has modulus $\sqrt{\alpha^2 + 4\pi^2}$. At the dominant pole α, the simple-pole residue formula gives

$$\mathrm{Res}(A;\alpha) \;=\; \frac{1}{-e^{\alpha}} \;=\; -\frac{1}{2}.$$

For any fixed $\alpha < R < \sqrt{\alpha^2 + 4\pi^2}$, Corollary 4 therefore gives

$$[z^n]A(z) \;=\; \frac{1}{2\alpha^{n+1}} + \mathcal{O}(R^{-n}), \qquad a_n \;\sim\; \frac{n!}{2(\log 2)^{n+1}}.$$

The radius $\log 2$ predicts the exponential scale of $a_n/n!$; the residue supplies the constant that completes the estimate.

Example 6. Consider permutations with no cycles of length 1 or 2. Removing those cycle lengths from the labelled set-of-cycles construction gives the EGF

$$B(z) \;=\; \exp\left(\sum_{j\geq 3} \frac{z^j}{j}\right) \;=\; \frac{e^{-z-z^2/2}}{1-z}.$$

The series expression is valid for $|z| < 1$, while the quotient gives an analytic continuation to $\mathbb{C} \setminus \{1\}$ and a meromorphic function on $\mathbb{C}$. Its only pole is the simple pole at 1. As in Lecture 6's derangement example, the numerator is entire and nonzero at the pole. The residue is

$$\mathrm{Res}(B;1) \;=\; \lim_{z\to 1}(z-1)\frac{e^{-z-z^2/2}}{1-z} \;=\; -\,e^{-3/2}.$$

There are no further poles, so Corollary 4 gives, for every fixed $R > 1$,

$$[z^n]B(z) \;=\; e^{-3/2} + \mathcal{O}(R^{-n}).$$

Since B is an EGF, $[z^n]B(z)$ is the number of these permutations divided by $n!$. Thus their proportion among all permutations tends to $e^{-3/2} \approx 22.31\%$.

Exercise 7. Let J be a fixed finite set of positive integers. Derive the EGF for permutations with no cycle length in J, then use its pole at 1 to show that their limiting proportion is $\exp\left(-\sum_{j\in J} \frac{1}{j}\right)$.

Example 8. The EGF for alternating permutations from Lecture 4 is

$$A(z) \;=\; \sec z + \tan z \;=\; \frac{1+\sin z}{\cos z}, \qquad a_n \;=\; n![z^n]A(z).$$

The denominator vanishes at $\pi/2 + k\pi$ for $k \in \mathbb{Z}$, but not every such point is a pole. At $z = -\pi/2$ and, more generally, wherever $\sin z = -1$, the identity

$$\frac{1+\sin z}{\cos z} \;=\; \frac{\cos z}{1-\sin z}$$

provides a holomorphic extension, since the denominator on the right is nonzero. The true poles are $\pi/2 + 2k\pi$ for $k \in \mathbb{Z}$; each is simple. Thus $\rho = \pi/2$ is the unique nearest pole, and the next nearest is

$-3\pi/2$. At $\rho = \pi/2$, the residue is

$$\mathrm{Res}(A;\rho) \;=\; \frac{1+\sin\rho}{-\sin\rho} \;=\; -2.$$

For any fixed $\pi/2 < R < 3\pi/2$, Corollary 4 gives

$$[z^n]A(z) \;=\; 2\left(\frac{2}{\pi}\right)^{n+1} + \mathcal{O}(R^{-n}), \qquad a_n \;\sim\; 2n!\left(\frac{2}{\pi}\right)^{n+1}.$$

Checking the numerator was essential: treating $-\pi/2$ as a second dominant pole would give an incorrect oscillating term.

Exercise 9. In Lecture 4, we found that the EGF for permutations with no three consecutive entries in decreasing order is

$$f(z) \;=\; \frac{\sqrt{3}}{2}\frac{e^{z/2}}{\cos\left(\frac{\sqrt{3}}{2}z+\frac{\pi}{6}\right)}.$$

Determine the asymptotic growth of f_n, the number of such permutations, as follows:

(a) Locate the poles of $f(z)$ by setting the denominator to zero, and show that the dominant pole is $\alpha_0 = \frac{2\pi}{3\sqrt{3}}$.

(b) Compute the residue of $f(z)$ at $z = \alpha_0$. Then apply Corollary 4 to deduce that

$$f_n \;\sim\; e^{\pi/(3\sqrt{3})}\left(\frac{3\sqrt{3}}{2\pi}\right)^{n+1} n!.$$

Rational Generating Functions and Exact Formulas

For rational generating functions, all pole contributions can be combined into an exact coefficient formula. The following examples illustrate exponential growth and polynomial growth with periodic corrections.

Example 10. Consider the sequence of Fibonacci numbers F_n, which satisfies $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0$ and $F_1 = 1$. Its ordinary generating function (OGF) is

$$F(z) \;=\; \sum_{n=0}^{\infty} F_n z^n \;=\; \frac{z}{1-z-z^2}.$$

The denominator has two simple roots, $\alpha = (-1+\sqrt{5})/2$ and $\beta = (-1-\sqrt{5})/2$. The residues are

$$\mathrm{Res}(F;\alpha) \;=\; \frac{\alpha}{-1-2\alpha} \;=\; -\frac{\alpha}{\sqrt{5}}, \qquad \mathrm{Res}(F;\beta) \;=\; \frac{\beta}{-1-2\beta} \;=\; \frac{\beta}{\sqrt{5}}.$$

A simple pole ρ has normalized principal-part coefficient $-\mathrm{Res}(F;\rho)/\rho$. Since F is rational with numerator degree smaller than denominator degree, Theorem 2 gives

$$F_n \;=\; \frac{1}{\sqrt{5}}\left(\alpha^{-n}-\beta^{-n}\right) \;=\; \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

Since $|(1+\sqrt{5})/2| > 1$ and $|(1-\sqrt{5})/2| < 1$, the second term decays to 0, and the first term dominates.

Exercise 11. Let w_n count words of length n over $\{a, b, c, d\}$ without four consecutive identical letters. Their OGF is

$$W(z) \;=\; \frac{1 + z + z^2 + z^3}{1 - 3z - 3z^2 - 3z^3}.$$

Let $\rho \approx 0.25308$ be the unique positive root of the denominator. Prove that ρ is simple and that every other zero has larger modulus. Compute $\operatorname{Res}(W; \rho)$ and apply Corollary 4 to show

$$w_n \;\sim\; \frac{4}{9(1 + 2\rho + 3\rho^2)}\,\rho^{-n-1} \;=\; C\rho^{-n}, \qquad C \approx 1.034, \quad \frac{1}{\rho} \approx 3.951.$$

The next example combines poles of different orders. The higher-order residue formula from Lecture 6 lets us calculate their contributions directly.

Example 12. Let $p_{\{1,2,4\}}(n)$ count integer partitions of n whose parts are restricted to 1, 2 and 4. Its OGF is

$$P(z) \;=\; \frac{1}{(1-z)(1-z^2)(1-z^4)} \;=\; \frac{1}{(1-z)^3(1+z)^2(1+z^2)}.$$

The poles are 1 of order 3, -1 of order 2 and $\pm i$ of order 1. Since P has numerator degree smaller than denominator degree, Theorem 2 gives its coefficients exactly from these four principal parts. To compute their contributions as residues, expand z^{-n-1} about each pole ρ. A principal-part term $c(1 - z/\rho)^{-j}$ contributes residue $-c\binom{n+j-1}{j-1}\rho^{-n}$ to $P(z)/z^{n+1}$, the negative of its coefficient contribution. Hence

$$p_{\{1,2,4\}}(n) \;=\; -\sum_{\rho \in \{1,-1,i,-i\}} \operatorname{Res}\left(\frac{P(z)}{z^{n+1}}; \rho\right).$$

Write $G_n(z) = \frac{P(z)}{z^{n+1}}$. The residue formula for a pole of order m from Lecture 6 gives

$$\begin{aligned}
\operatorname{Res}(G_n; 1) \;&=\; \frac{1}{2}\,\frac{\mathsf{d}^2}{\mathsf{d}z^2}\left(-\frac{1}{z^{n+1}(1+z)^2(1+z^2)}\right)\Big|_{z=1} \;=\; -\frac{2n^2 + 14n + 21}{32},\\
\operatorname{Res}(G_n; -1) \;&=\; \frac{\mathsf{d}}{\mathsf{d}z}\left(\frac{1}{z^{n+1}(1-z)^3(1+z^2)}\right)\Big|_{z=-1} \;=\; -\frac{(-1)^n(2n+7)}{32},\\
\operatorname{Res}(G_n; i) \;&=\; \frac{1+i}{16 i^{n+2}}, \qquad \operatorname{Res}(G_n; -i) \;=\; \frac{1+i}{16(-i)^{n+1}}.
\end{aligned}$$

The two non-real residues combine as

$$\operatorname{Res}(G_n; i) + \operatorname{Res}(G_n; -i) \;=\; \begin{cases} -\frac{1}{8}, & n \equiv 0, 1 \pmod 4,\\ \frac{1}{8}, & n \equiv 2, 3 \pmod 4. \end{cases}$$

Substituting these residues into the coefficient formula yields

$$p_{\{1,2,4\}}(n) \;=\; \begin{cases} (m+1)^2, & n = 4m \text{ or } n = 4m+1,\\ (m+1)(m+2), & n = 4m+2 \text{ or } n = 4m+3. \end{cases}$$

The pole of order 3 at 1 contributes the quadratic polynomial $(2n^2 + 14n + 21)/32$. The pole of order 2 at -1 contributes a linear term whose sign alternates, and the two simple poles contribute bounded periodic corrections. In particular, $p_{\{1,2,4\}}(n) \sim n^2/16$, but the exact formula retains the fluctuations that the leading estimate misses.

The partition formula above is polynomial on each residue class modulo 4. This pattern comes from its poles at roots of unity. More generally, partitions with parts in a fixed finite set S have generating function $\prod_{s\in S}(1-z^s)^{-1}$, whose poles are all roots of unity. The following consequence of Theorem 2 explains the resulting structure.

Corollary 13. If every pole of a rational function holomorphic at 0 is a root of unity, its coefficient sequence is eventually quasipolynomial: for some integer $q \geq 1$ and polynomials $P_0, \ldots, P_{q-1}$, the coefficient at n equals $P_r(n)$ for all sufficiently large $n \equiv r \pmod{q}$. If the numerator degree is smaller than the denominator degree, the equality holds for every $n \geq 0$.

Proof: Choose q such that $\rho^q = 1$ for every pole ρ. On each residue class modulo q, every factor ρ^{-n} is constant, while $\binom{n+j-1}{j-1}$ is polynomial in n. Apply Theorem 2; the polynomial part Q affects only finitely many coefficients. □

Exercise 14. Let $S = \{s_1, \ldots, s_k\}$ be a finite set of positive integers with $1 \in S$ and $k \geq 2$, and let $p_S(n)$ count integer partitions of n whose parts all lie in S. Its OGF is

$$P_S(z) \;=\; \prod_{s\in S} \frac{1}{1-z^s}, \qquad p_S(n) \;=\; [z^n]P_S(z).$$

(a) Show that the pole at $z = 1$ has order k, while every other pole has order at most $k-1$.

(b) Deduce the first-order asymptotic

$$p_S(n) \;\sim\; \frac{1}{(k-1)!\prod_{s\in S} s} \cdot n^{k-1}.$$

(c) If $\gcd(S \setminus \{1\}) = 1$, show that

$$p_S(n) \;=\; \frac{1}{(k-1)!\prod_{s\in S} s} \cdot n^{k-1} \;+\; \frac{\left(\sum_{s\in S} s\right)}{2(k-2)!\prod_{s\in S} s} \cdot n^{k-2} \;+\; o(n^{k-2}).$$

The local form of each dominant pole determines its coefficient contribution, while the distance to the next singularities controls the error. These methods apply when the generating function is meromorphic on a neighborhood of the chosen closed disk. The tree function and dilogarithm in Lecture 6 can be analytically continued beyond their convergence disks, but their singularities at $1/e$ and 1, respectively, are not poles. Lecture 8 develops singularity analysis for such algebraic and logarithmic singularities.