L8: Singularity Analysis for Algebraic and Logarithmic Singularities

Analytic Combinatorics
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Lecture 7 extracted coefficients from the dominant poles of generating functions. We now turn to algebraic and logarithmic singularities, whose canonical examples are $\sqrt{z}$ and $\log z$. The new difficulty is that their analytic continuations can be *multi-valued*. For $z \neq 0$, the equation $y^2 = z$ has two distinct solutions, y and $-y$. For real $x > 0$, the convention $\sqrt{x} > 0$ consistently selects one of the two square roots, giving a differentiable function of x. In the complex plane, we also require the chosen values to vary analytically with z. We can make such a choice locally, but continuation once around 0 changes its sign; see Example 1 for details. The logarithm presents the same difficulty: if $e^y = z$, then $y + 2k\pi i$ is also a logarithm of z for every $k \in \mathbb{Z}$, and continuation around 0 can change the chosen value. Thus no pointwise convention gives a single-valued analytic choice for either function throughout the complex plane.

If continuation around a singular point returns a different value at the starting point, that singular point is called a branch point. To maintain a consistent analytic choice, we can remove a ray extending from the branch point to infinity, such as the negative real axis $(-\infty, 0]$ for $\sqrt{z}$ and $\log z$. Such a cut is called a branch cut. The tree function and dilogarithm from Lecture 6 also have branch points, at $1/e$ and 1 respectively.

Cauchy's coefficient formula remains valid inside the convergence disk, where the generating function is single-valued. To deform the contour beyond that disk, we must stay in a domain where the chosen continuation is single-valued and analytic; we cannot treat a branch point as a pole and simply add a residue contribution. To extract coefficient asymptotics from these singularities, we apply singularity analysis, a method introduced by Flajolet and Odlyzko [FO90]. The method has three stages:

- **Local Expansion**: Scale the dominant singularity to $z = 1$ and establish an expansion $f(z) = c(1-z)^{-\alpha} + \mathcal{O}((1-z)^{-\beta})$ for some $\beta < \alpha$ and constant c.
- **Transfer**: Apply the *Transfer Theorem* (Theorem 9) to relate $[z^n]f(z)$ to the coefficients of the leading term $c(1-z)^{-\alpha}$.
- **Standard Scale**: Evaluate these coefficients asymptotically using the *Standard Scale Theorem* (Theorem 7), whose proof uses the Gamma function and *Hankel contours*.

This lecture builds the needed tools before applying them, in the following order:

1. **Branches and contours.** We explain branch points and branch cuts, then introduce Δ-domains and Hankel contours to describe where and how Cauchy's contour can be deformed.
2. **Algebraic coefficient estimates.** We develop the standard-scale and transfer theorems, with applications to central binomial coefficients and Catalan numbers.
3. **Implicit equations and trees.** We use the characteristic equation to locate singularities of $y = z\phi(y)$ and derive square-root expansions, explaining the recurring $n^{-3/2}$ factor in tree counts.
4. **Logarithms and expected values.** We extend the standard scale to logarithmic factors, then use bivariate generating functions to estimate cycle counts in permutations, component counts in 2-regular graphs and average depths in binary trees.

Algebraic and Logarithmic Singularities, Branch Points and Branch Cuts

We begin by examining why analytic continuation around a singular point can change the value of a square root or logarithm.

Example 1. Consider the complex square root function $f(z) = \sqrt{z}$. Using polar coordinates, any $z \in \mathbb{C} \setminus \{0\}$ can be written as $z = re^{i\theta}$. If we define the square root as $\sqrt{z} = \sqrt{r}e^{i\theta/2}$, what happens when we trace a continuous path in a circle around the origin? Starting at $z = 1$ (where $r = 1, \theta = 0$), we have $\sqrt{1} = 1$. As we travel counterclockwise along the unit circle, θ increases to 2π. After one turn, the continued value is $e^{i\pi} = -1$. We have returned to $z = 1$, but the function value has changed.

For the logarithm, choose the initial value 0 at $z = 1$. Along the same path $z = e^{i\theta}$, the continued value is $i\theta$, since its exponential equals z. After one turn, z returns to 1 but the logarithm reaches $2\pi i$.

In contrast, $1/z = e^{-i\theta}$ returns to 1 after one turn. Its pole at 0 prevents a holomorphic extension through that point, but the function remains single-valued around it. The algebraic and logarithmic branch points above introduce an additional difficulty: continuation around the singularity can change the function value.

Thus neither square root nor logarithm admits a single-valued analytic branch on $\mathbb{C} \setminus \{0\}$.

Definition 2. A branch point is a point around which continuation of a local determination along a sufficiently small loop winding once can return a different value. A branch is a single-valued analytic choice of values on a domain. A branch cut is a curve removed from the domain to make such a choice possible. For $\sqrt{z}$ and $\log z$, removing a ray from 0 to infinity suffices.

Example 3. The branch cut for the functions in Example 1 is not unique. Removing $(-\infty, 0]$ lets us choose $-\pi < \theta < \pi$ in $z = re^{i\theta}$, giving the *principal branches* $\sqrt{z} = \sqrt{r}e^{i\theta/2}$ and $\log z = \log r + i\theta$. Alternatively, removing $[0, \infty)$ permits $0 < \theta < 2\pi$, and the same formulas define analytic branches on this different domain. For example, at $z = -i$, the logarithm is $-\pi i/2$ with the first choice and $3\pi i/2$ with the second. Both cuts prevent a full loop around 0.

In combinatorial generating functions, singularities often manifest as terms like $(1 - z/\rho)^{-\alpha}$ for non-integer α. The branch point is $z = \rho$, and the conventional branch cut is the ray extending outwards from the singularity to infinity: $[\rho, +\infty)$.

Δ-Domains and Hankel Contours

Once a branch is fixed, we return to Cauchy's coefficient formula. We need a region in which to deform the contour, a path that avoids the branch cut and a way to evaluate the resulting integral. The Δ-domain, Hankel contour and reciprocal Gamma function will provide these three ingredients.

Example 4. Consider $f(z) = (1 - z)^{-1/2}$, choosing the branch with $f(0) = 1$ and cut $[1, \infty)$. For a positively oriented circle γ centered at 0 with radius less than 1, Cauchy's formula gives

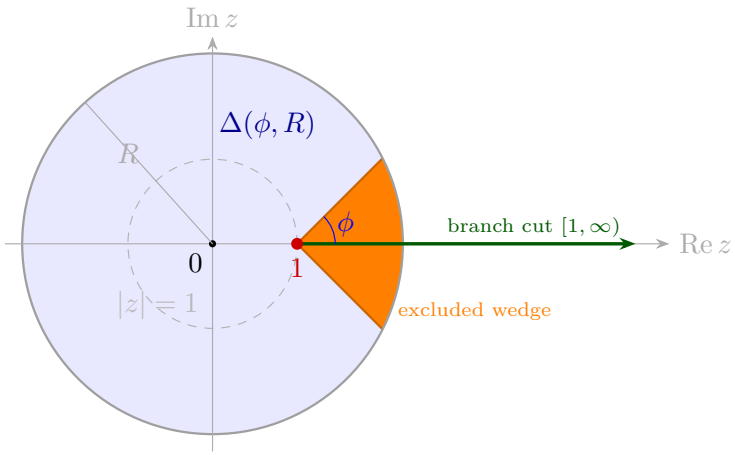
$$[z^n]f(z) \;=\; \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z^{n+1}}\, \mathsf{d}z.$$

Where $|z| > 1$, the factor z^{-n-1} decays exponentially with n. We would therefore like to push the whole contour outside the unit circle. However, this is not allowed, as any closed contour enclosing 0 and lying entirely outside the unit circle must cross the branch cut. As we shall see, the compromise is to push most of the contour outside the unit circle, leaving only a small detour near $z = 1$.

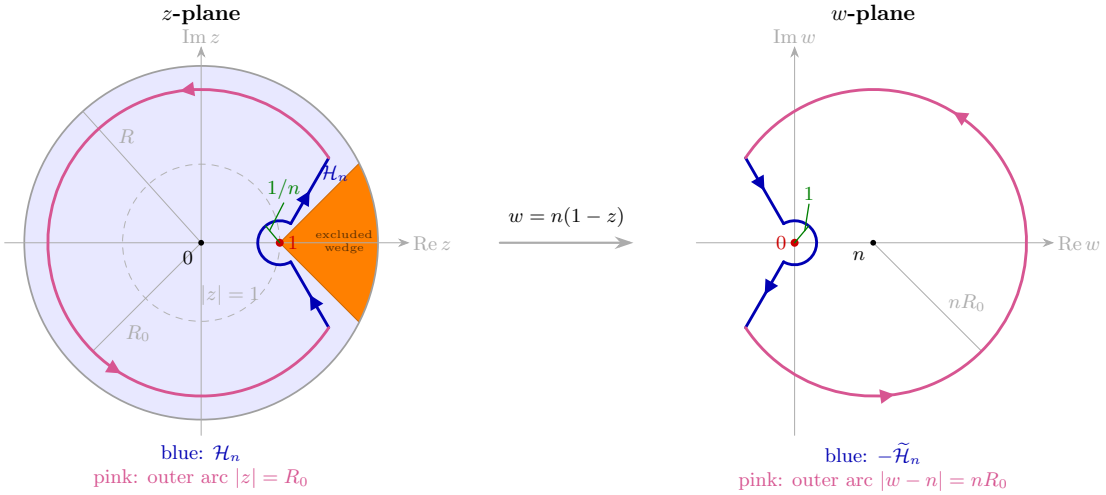
Definition 5. Given $\phi \in (0, \pi/2)$ and $R > 1$, a Δ-domain is the open set

$$\Delta(\phi, R) \;=\; \{z \in \mathbb{C} \;\mid\; |z| < R,\; z \neq 1,\; |\arg(z-1)| > \phi\}.$$

A function is Δ-analytic if it is analytic in some such domain.



This dented disk contains the open unit disk but excludes $z = 1$ and a wedge around $[1, \infty)$. For a dominant singularity at $\rho > 0$, we first scale it to 1: f is Δ-analytic at ρ when $f(\rho z)$ is analytic in a Δ-domain.



The two panels use different coordinate scales.

For f analytic in $\Delta(\phi, R)$, choose $\phi < \psi < \pi/2$ and $1 < R_0 < R$. Let $\mathcal{H}_n$ denote the blue contour in the z-plane of the figure above, which runs from $|z| = R_0$ towards 1 along the ray $z = 1 + te^{-i\psi}$, rounds 1 on the left at distance $1/n$ and returns to $|z| = R_0$ along $z = 1 + te^{i\psi}$. Adding the long pink arc of $|z| = R_0$ closes the contour around 0. This outer arc contributes $\mathcal{O}(R_0^{-n})$ to the coefficient integral because f is bounded there. Under $w = n(1 - z)$, the blue contour on the left maps to the blue contour on the right. Define $\widetilde{\mathcal{H}}_n$ by reversing the orientation of this image, so its small arc is traversed counterclockwise. This reversal cancels the minus sign in $\mathtt{d}z = -\mathtt{d}w/n$. Both $\mathcal{H}_n$ and $\widetilde{\mathcal{H}}_n$ exclude the pink outer arcs. Extending the slanted sides of $\widetilde{\mathcal{H}}_n$ to infinity gives a Hankel contour $\mathcal{H}$.

The radius $1/n$ matches the scale on which z^{-n-1} changes near $z = 1$. In the scaled coordinates, the cut becomes the negative real axis and the small arc has radius 1. For the model $(1-z)^{-\alpha}$ and fixed w, we have

$$(1-z)^{-\alpha} \;=\; n^{\alpha} w^{-\alpha}, \qquad |\mathsf{d}z| \;=\; \frac{|\mathsf{d}w|}{n}, \qquad z^{-n-1} \;=\; \Big(1-\frac{w}{n}\Big)^{-n-1} \;\longrightarrow\; e^{w}.$$

Together, these factors give the prefactor $n^{\alpha-1}$ and the limiting integrand $e^{w}w^{-\alpha}$. As n grows, the endpoints of $\widetilde{\mathcal{H}}_n$ recede to infinity along $\mathcal{H}$. This suggests that the leading constant is given by Hankel's integral representation of the reciprocal Gamma function [FS09, Theorem B.1].

Theorem 6. Hankel's formula. For every $\alpha \in \mathbb{C}$,

$$\frac{1}{\Gamma(\alpha)} \;=\; \frac{1}{2\pi i}\int_{\mathcal{H}} e^{w} w^{-\alpha}\,\mathsf{d}w \;=\; \lim_{n\to\infty} \frac{1}{2\pi i}\int_{\widetilde{\mathcal{H}}_n} e^{w} w^{-\alpha}\,\mathsf{d}w.$$

Here $1/\Gamma$ denotes its entire continuation, vanishing at $0, -1, \ldots$. The contour $\mathcal{H}$ has the orientation described above, and its integral is improper at infinity. Cauchy's theorem and exponential decay in the left half-plane allow the usual horizontal Hankel paths to be deformed into these rays. Use the *principal logarithm*: for $w = re^{i\theta}$ with $r > 0$ and $-\pi < \theta < \pi$, set

$$\operatorname{Log} w \;=\; \log r + i\theta, \qquad w^{-\alpha} \;=\; \exp\big(-\alpha \operatorname{Log} w\big).$$

The factor $1/\Gamma(\alpha)$ supplies the multiplicative constant in the coefficient asymptotics of a power singularity. In Example 4, $\alpha = 1/2$ gives $1/\Gamma(1/2) = 1/\sqrt{\pi}$, leading to the estimate $1/\sqrt{\pi n}$. The next section makes this calculation precise through Theorem 7.

The Standard Scale

We now study functions of the form $(1-z)^{-\alpha}$. By the generalized binomial theorem, for any non-negative integer n and $\alpha \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$, we have the identity

$$[z^n](1-z)^{-\alpha} \;=\; \binom{n+\alpha-1}{n} \;=\; \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}. \tag{1}$$

The Hankel-contour argument below gives the leading term; a refined Stirling formula supplies the correction.

Theorem 7. Standard Scale [FS09, Theorem VI.1, Eq. (13)]. For any $\alpha \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}$, we have

$$[z^n](1-z)^{-\alpha} \;=\; \frac{n^{\alpha-1}}{\Gamma(\alpha)}\left(1 \;+\; \frac{\alpha(\alpha-1)}{2n} \;+\; \mathcal{O}\left(\frac{1}{n^2}\right)\right).$$

Proof: By Cauchy's coefficient formula, deforming a small positively oriented circle into $\mathcal{H}_n$ and the outer arc $|z| = R_0$ gives

$$[z^n](1-z)^{-\alpha} \;=\; \frac{1}{2\pi i}\int_{\mathcal{H}_n} \frac{(1-z)^{-\alpha}}{z^{n+1}}\,\mathsf{d}z + \mathcal{O}(R_0^{-n}).$$

The outer-arc estimate follows because $(1-z)^{-\alpha}$ is bounded there. Substitute $z = 1 - w/n$ and reverse the image contour to obtain $\widetilde{\mathcal{H}}_n$. This reversal cancels the minus sign in $\mathsf{d}z = -\mathsf{d}w/n$, so

$$[z^n](1-z)^{-\alpha} \;=\; \frac{n^{\alpha-1}}{2\pi i}\int_{\widetilde{\mathcal{H}}_n} w^{-\alpha}\Big(1-\frac{w}{n}\Big)^{-n-1}\,\mathsf{d}w + \mathcal{O}(R_0^{-n}).$$

The contour is still finite and depends on n. On every bounded portion of $\mathcal{H}$, the kernel $(1-w/n)^{-n-1}$ converges uniformly to e^w. To justify passing to the infinite contour, we also need a bound on the tails.

Let ℓ be the fixed distance from $z=1$ to either endpoint of $\mathcal{H}_n$. On the scaled rays, $w = te^{\pm i(\pi-\psi)}$ with $1 \le t \le n\ell$. Since $\cos\psi > 0$, we have

$$\left|1-\frac{w}{n}\right|^{-n-1} \;\le\; \left(1+\frac{t\cos\psi}{n}\right)^{-n-1} \;\le\; e^{-\kappa t}, \qquad \kappa \;=\; \frac{\cos\psi}{1+\ell\cos\psi} \;>\; 0. \tag{2}$$

The last inequality follows from $\log(1+x) \ge x/(1+x)$ for $x \ge 0$. The bound $t^{-\alpha}e^{-\kappa t}$ is integrable on $[1,\infty)$, and the fixed inner arc stays away from $w=0$. Dominated convergence therefore gives

$$\lim_{n\to\infty} \frac{1}{2\pi i}\int_{\widetilde{\mathcal{H}}_n} w^{-\alpha}\left(1-\frac{w}{n}\right)^{-n-1}\,\mathsf{d}w \;=\; \frac{1}{2\pi i}\int_{\mathcal{H}} w^{-\alpha}e^{w}\,\mathsf{d}w \;=\; \frac{1}{\Gamma(\alpha)},$$

where the last equality is Theorem 6. The outer-arc error is exponentially small, so this proves $[z^n](1-z)^{-\alpha} \sim n^{\alpha-1}/\Gamma(\alpha)$.

To obtain the correction term, we need Stirling's formula with a remainder smaller than $1/x$. Its logarithmic form is [FS09, Appendix B.3, p. 747]

$$\log\Gamma(x) \;=\; \left(x-\frac{1}{2}\right)\log x - x + \frac{1}{2}\log(2\pi) + \frac{1}{12x} + \mathcal{O}(x^{-2}) \qquad (x \to +\infty).$$

Apply this formula at $x = n+\alpha$ and $x = n+1$. Taylor expansion in $1/n$, followed by subtraction and exponentiation, gives the stated correction term and error bound via Equation (1). □

Example 8. The central binomial coefficients are generated by

$$\sum_{n\ge 0}\binom{2n}{n}z^n \;=\; \frac{1}{\sqrt{1-4z}}.$$

Scaling the singularity to 1 gives

$$[z^n](1-4z)^{-1/2} \;=\; 4^n[w^n](1-w)^{-1/2}.$$

Using Theorem 7 with $\alpha = 1/2$ yields the two-term estimate

$$\binom{2n}{n} \;=\; \frac{4^n}{\sqrt{\pi n}}\left(1-\frac{1}{8n}+\mathcal{O}(n^{-2})\right).$$

The leading term comes directly from the square-root singularity at $z = 1/4$; the correction uses Stirling's formula through our proof of Theorem 7.

The Transfer Theorem

The standard scale estimates coefficients of model singularities. The Transfer Theorem extends these estimates to functions with a controlled remainder near the singularity.

For a Δ-analytic function, the deformed contour approaches 1 along rays outside the excluded wedge, with an inner arc of radius $1/n$. Every fixed portion away from 1 lies outside the unit circle and contributes exponentially little. This geometry turns local bounds on the remainder into coefficient bounds. A local estimate alone is insufficient: another dominant singularity can change the answer, as Example 19 illustrates.

Theorem 9. Transfer Theorem [FS09, Theorems VI.1 and VI.3]. Assume that $f(z)$ is Δ-analytic. Let $\alpha, \beta \in \mathbb{R}$ with $\alpha \notin \{0, -1, -2, \dots\}$ and $\beta < \alpha$. If, as $z \to 1$ inside the Δ-domain,

$$f(z) \;=\; c(1-z)^{-\alpha} \;+\; \mathcal{O}\Big((1-z)^{-\beta}\Big),$$

then the coefficients satisfy

$$[z^n]f(z) \;=\; c[z^n](1-z)^{-\alpha} + \mathcal{O}(n^{\beta-1}) \;=\; \frac{c}{\Gamma(\alpha)} n^{\alpha-1} + \mathcal{O}\Big(n^{\alpha-2} + n^{\beta-1}\Big).$$

Informal Proof: Separate the model term by setting $g(z) = f(z) - c(1-z)^{-\alpha}$. Then g is Δ-analytic and $g(z) = \mathcal{O}(|1-z|^{-\beta})$ near 1. Deform the coefficient contour inside the Δ-domain as before, keeping the outer arc at a fixed radius $1 < R_0 < R$. With $z = 1 - w/n$ and the standard orientation of $\widetilde{\mathcal{H}}_n$, we obtain

$$[z^n]g(z) \;=\; \frac{1}{2\pi i n} \int_{\widetilde{\mathcal{H}}_n} g\Big(1 - \frac{w}{n}\Big) \Big(1 - \frac{w}{n}\Big)^{-n-1} \, \mathsf{d}w + \mathcal{O}(R_0^{-n}).$$

The local bound on g, extended along the fixed ray segments by compactness, gives $|g(1-w/n)| \le Cn^{\beta}|w|^{-\beta}$ on $\widetilde{\mathcal{H}}_n$. On the rays, use the uniform bound $e^{-\kappa t}$ from Equation (2); on the fixed unit arc, the kernel is uniformly bounded for large n. Hence, for a constant C' independent of n,

$$|[z^n]g(z)| \;\le\; C' n^{\beta-1} \left(1 + \int_1^{\infty} t^{-\beta} e^{-\kappa t} \, \mathsf{d}t\right) + \mathcal{O}(R_0^{-n}) \;=\; \mathcal{O}(n^{\beta-1}).$$

This proves the estimate relative to the exact model coefficients. Replacing those coefficients by their first-order asymptotic introduces the additional error $\mathcal{O}(n^{\alpha-2})$ from Theorem 7. □

Example 10. The OGF of the Catalan numbers is $C(z) = (1 - \sqrt{1-4z})/(2z)$, with dominant singularity $1/4$. Rescale to $f(w) = C(w/4) = 2(1 - \sqrt{1-w})/w$. This function is analytic on $\mathbb{C} \setminus [1, \infty)$, with a removable singularity at $w = 0$, so it is Δ-analytic. We expand $f(w)$ around $w = 1$. Since $1/w = 1 + (1-w) + (1-w)^2 + \dots$, we have

$$f(w) \;=\; 2(1 - \sqrt{1-w})(1 + \mathcal{O}(1-w)) \;=\; 2 \;-\; 2(1-w)^{1/2} \;+\; \mathcal{O}(1-w).$$

By the change of variables, $C_n = [z^n]C(z) = 4^n[w^n]f(w)$. Apply Theorem 9 to $f(w) - 2$, with $c = -2$, $\alpha = -1/2$ and $\beta = -1$; the constant contributes no coefficients for $n \ge 1$.

$$C_n \;=\; 4^n \Big(-2[w^n](1-w)^{1/2} + \mathcal{O}(n^{-2})\Big) \;\sim\; -\frac{2 \cdot 4^n}{\Gamma(-1/2)} n^{-3/2} \;=\; \frac{4^n}{\sqrt{\pi}\, n^{3/2}}.$$

Implicit Functions and the Characteristic Equation

In Lecture 5, we studied combinatorial classes defined by implicit equations of the form $y(z) = z\phi(y(z))$, such as various families of trees. We will see that implicit equations typically induce square-root branch points. By determining the location and local behaviour of these singularities, we can apply singularity analysis to establish universal asymptotic laws for tree-like structures.

Example 11. For rooted labelled trees, Lecture 5 gave the equation $T(z) = ze^{T(z)}$. Differentiating implicitly gives

$$T'(z) \;=\; e^{T(z)} \;+\; ze^{T(z)}T'(z), \qquad T'(z) \;=\; \frac{e^{T(z)}}{1 - ze^{T(z)}}.$$

We look for a positive real singularity ρ where $T(z)$ approaches a finite value τ as $z \uparrow \rho$, while $T'(z)$ diverges. Taking limits in $T(z) = ze^{T(z)}$ gives the first equation below. Since the numerator of $T'(z)$ tends to $e^{\tau} \neq 0$, divergence requires its denominator to tend to zero, giving the second equation:

$$\tau \;=\; \rho e^{\tau}, \qquad 1 \;=\; \rho e^{\tau}.$$

Comparing the two equations gives $\tau = 1$, and substitution then gives $\rho = 1/e$.

To verify this behaviour, rewrite the defining equation as $z = T(z)e^{-T(z)}$. The function $t \mapsto te^{-t}$ is strictly increasing from 0 to $1/e$ on $[0, 1]$, so its inverse branch with $T(0) = 0$ satisfies $T(z) \uparrow 1$ as $z \uparrow 1/e$. The derivative formula therefore gives $T'(z) \to +\infty$. Since the Taylor series of T has radius $1/e$, as noted in Lecture 6, this is indeed a dominant singularity.

Definition 12. A formal power series $y(z)$ is said to be defined by a tree-like implicit equation if $y(z) = z\phi(y(z))$, where $\phi(y)$ is a power series with non-negative coefficients, $\phi(0) \neq 0$, and $\phi(y)$ is non-linear (i.e., $\phi''(y) \not\equiv 0$).

For a tree-like equation $y = z\phi(y)$ with ϕ analytic near 0, implicit differentiation gives

$$y'(z) \;=\; \frac{\phi(y(z))}{1 - z\phi'(y(z))}.$$

Suppose $y(z) \to \tau$ as real $z \uparrow \rho > 0$, with τ strictly inside the convergence disk of ϕ. By the analytic implicit function theorem, a singularity at ρ then requires $1 - \rho\phi'(\tau) = 0$, producing a vertical tangent. Such an interior critical point need not exist: $y(z)$ may instead reach the convergence boundary of ϕ.

Definition 13. For $y(z) = z\phi(y(z))$, the characteristic system for an interior critical point (ρ, τ) with $\rho, \tau > 0$ is:

$$\tau \;=\; \rho\phi(\tau) \quad \text{and} \quad 1 \;=\; \rho\phi'(\tau).$$

Eliminating ρ gives the characteristic equation:

$$\phi(\tau) - \tau\phi'(\tau) \;=\; 0.$$

Example 14. Plane rooted trees, where each node carries an ordered sequence of children, satisfy

$$P(z) \;=\; \frac{z}{1 - P(z)}.$$

Here $\phi(y) = (1 - y)^{-1}$. The characteristic equation is

$$\phi(\tau) - \tau\phi'(\tau) \;=\; \frac{1}{1-\tau} - \frac{\tau}{(1-\tau)^2} \;=\; \frac{1 - 2\tau}{(1-\tau)^2} \;=\; 0.$$

Thus $\tau = 1/2$ and $\rho = \tau/\phi(\tau) = 1/4$. This locates the Catalan-type singularity at $1/4$; Theorem 15 gives its square-root form.

The Square-Root Singularity and Universal Tree Asymptotics

To apply singularity analysis, we must determine the local expansion of $y(z)$ around $z = \rho$. We expand the relation $z(y) = y/\phi(y)$ around $y = \tau$ using a Taylor series, and then invert it to find $y(z)$ around $z = \rho$.

Theorem 15. Smooth Implicit-Function Theorem [FS09, Theorem VI.6]. Let $y(z) = z\phi(y(z))$ be a tree-like implicit equation, and let $R_\phi > 0$ be the convergence radius of ϕ. Suppose $0 < \tau < R_\phi$ satisfies $\phi(\tau) - \tau\phi'(\tau) = 0$, and set $\rho = \tau/\phi(\tau)$. Then y has a branch point at ρ and continues to a slit disk $D(\rho, \varepsilon) \setminus [\rho, \rho + \varepsilon)$, where, uniformly as $z \to \rho$,

$$y(z) \;=\; \tau \;-\; \sqrt{\frac{2\phi(\tau)}{\phi''(\tau)}}\sqrt{1 - \frac{z}{\rho}} \;+\; \mathcal{O}\left(1 - \frac{z}{\rho}\right).$$

Proof: Set $h(t) = \phi(t) - t\phi'(t)$. Non-negative coefficients and nonlinearity give $h'(t) = -t\phi''(t) < 0$ for $0 < t < R_\phi$. Since $h(\tau) = 0$, the map $z(t) = t/\phi(t)$ has derivative $h(t)/\phi(t)^2 > 0$ on $[0, \tau)$ and increases from 0 to ρ. Its inverse is the branch with $y(0) = 0$, so $y(z) \uparrow \tau$ as $z \uparrow \rho$. The coefficients of y are non-negative, so Pringsheim's theorem excludes a smaller convergence radius.

We now expand $z(y) = y/\phi(y)$ around $y = \tau$. Its first derivative vanishes there, and its second derivative is

$$z''(\tau) \;=\; -\frac{\tau\phi''(\tau)}{\phi(\tau)^2} \;=\; -\frac{\rho\phi''(\tau)}{\phi(\tau)}.$$

The Taylor expansion of $z(y)$ around $y = \tau$ is thus:

$$z(y) \;=\; \rho \;+\; \frac{1}{2}z''(\tau)(y - \tau)^2 \;+\; \mathcal{O}((y - \tau)^3).$$

Substituting the value of $z''(\tau)$ gives:

$$z \;=\; \rho \;-\; \frac{\rho\phi''(\tau)}{2\phi(\tau)}(y - \tau)^2 \;+\; \mathcal{O}((y - \tau)^3).$$

Dividing by ρ and rearranging, we find:

$$1 - \frac{z}{\rho} \;=\; \frac{\phi''(\tau)}{2\phi(\tau)}(y - \tau)^2 \;+\; \mathcal{O}((y - \tau)^3).$$

Taking the square root (and noting that $y(z) < \tau$ for $z < \rho$ on the real positive axis) yields:

$$y(z) - \tau \;\sim\; -\sqrt{\frac{2\phi(\tau)}{\phi''(\tau)}}\sqrt{1 - \frac{z}{\rho}}.$$

Since the quadratic coefficient is non-zero, analytic inversion in $X = \sqrt{1 - z/\rho}$ gives a convergent Puiseux series, analytic on a sufficiently small slit disk about ρ and with a uniform $\mathcal{O}(X^2)$ remainder. Choose $X > 0$ for real $z < \rho$. This branch agrees with y just below ρ, so the identity theorem identifies it with y and any Δ-continuation near ρ. □

Applying the Transfer Theorem to the local square-root expansion gives the coefficient asymptotic below. For tree-like recursions satisfying the stated hypotheses, it has the *universal form* $C\rho^{-n}n^{-3/2}$: the exponent $-3/2$ comes from the square-root singularity. The proof is left as an exercise (Exercise 17).

Theorem 16. Universal Tree Asymptotics. Under the conditions of Theorem 15, suppose additionally that $y(z)$ is analytic in a scaled Δ-domain $\{z : z/\rho \in \Delta(\theta, R)\}$. Then

$$[z^n]y(z) \sim \sqrt{\frac{\phi(\tau)}{2\pi\phi''(\tau)}}\rho^{-n}n^{-3/2}.$$

Exercise 17. Prove Theorem 16 by applying the Transfer Theorem (Theorem 9) to the expansion in Theorem 15. *Hint.* Apply it to $f(w) = y(\rho w) - \tau$ and use $\Gamma(-1/2) = -2\sqrt{\pi}$.

Example 18. Let us return to the rooted labelled trees with EGF $T(z) = ze^{T(z)}$. We established in Example 11 that $\phi(y) = e^y$, $\tau = 1$ and $\rho = 1/e$. Also, $\phi''(\tau) = e$. By Lecture 6, T is analytic on $\mathbb{C} \setminus [1/e, \infty)$, providing a scaled Δ-domain and excluding other singularities on $|z| = 1/e$. Applying Theorem 16 gives

$$[z^n]T(z) \sim \sqrt{\frac{e}{2\pi e}}e^n n^{-3/2} = \frac{e^n}{\sqrt{2\pi}}n^{-3/2}.$$

Since $T(z)$ is an exponential generating function, the number of trees is $t_n = n![z^n]T(z)$. By Stirling's approximation, $n! \sim \sqrt{2\pi n}(n/e)^n$. This agrees asymptotically with Cayley's exact formula $t_n = n^{n-1}$:

$$t_n \sim \left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right)\left(\frac{e^n}{\sqrt{2\pi}}n^{-3/2}\right) = n^{n-1}.$$

Example 19. Complete ordered binary trees, with z marking all nodes, satisfy $B(z) = z(1 + B(z)^2)$. The characteristic system gives $\tau = 1$ and $\rho = 1/2$, so the positive singularity alone contributes $2^n n^{-3/2}/\sqrt{2\pi}$. However, the explicit form $B(z) = \frac{1-\sqrt{1-4z^2}}{2z}$ reveals another dominant singularity at $z = -1/2$. The coefficients are periodic: $[z^n]B(z) = 0$ for even n, while $[z^{2k+1}]B(z) = C_k$. The Catalan asymptotic therefore gives

$$[z^n]B(z) \sim \sqrt{\frac{2}{\pi}}2^n n^{-3/2} \quad \text{along odd } n.$$

The contributions add for odd n and cancel for even n. Every scaled Δ-domain at $1/2$ contains $-1/2$, so B fails the hypothesis of Theorem 16.

Example 20. Unary-binary (Motzkin) trees satisfy $U = z(1 + U + U^2)$ (Lecture 5). With $\phi(y) = 1 + y + y^2$, the characteristic equation $1 - \tau^2 = 0$ gives $\tau = 1$ and $\rho = 1/3$. The solution $U(z) = (1 - z - \sqrt{(1-3z)(1+z)})/(2z)$ has a removable singularity at 0 and branch points at $1/3$ and -1. Thus $1/3$ is the unique dominant singularity, with a scaled Δ-domain. Since $\phi(1) = 3$ and $\phi''(1) = 2$, Theorem 16 gives

$$[z^n]U(z) \sim \sqrt{\frac{3}{2\pi \cdot 2}}3^n n^{-3/2} = \sqrt{\frac{3}{4\pi}}3^n n^{-3/2}.$$

Example 21. Acyclic graphs are sets of unrooted labelled trees, so $\mathcal{F} = \mathrm{SET}(\mathcal{U})$ and $F(z) = \exp(U(z))$. Recall from Lecture 4 that $U(z) = T(z) - T(z)^2/2$, where $T(z) = ze^{T(z)}$ is the EGF of rooted labelled trees.

To expand near $\rho = 1/e$, set $x = 1 - ez$ and $\delta = T(z) - 1$. By Theorem 15, $\delta = -\sqrt{2}x^{1/2} + \mathcal{O}(x)$. Expanding $ez = (1+\delta)e^{-\delta}$ gives

$$\frac{\delta^2}{2} \;=\; x + \frac{\delta^3}{3} + \mathcal{O}(\delta^4) \;=\; x - \frac{2\sqrt{2}}{3}x^{3/2} + \mathcal{O}(x^2).$$

Since $U(z) = 1/2 - \delta^2/2$, we obtain

$$\begin{aligned} U(z) &= \frac{1}{2} - x + \frac{2\sqrt{2}}{3}x^{3/2} + \mathcal{O}(x^2), \\ F(z) &= \sqrt{e}\left(1 - x + \frac{2\sqrt{2}}{3}x^{3/2} + \mathcal{O}(x^2)\right). \end{aligned}$$

In both expansions, the first non-analytic term is proportional to $x^{3/2}$. These functions inherit the slit-plane continuation of T from Lecture 6. The polynomial terms contribute no coefficients for $n \geq 2$, so the Transfer Theorem (Theorem 9), with $\Gamma(-3/2) = 4\sqrt{\pi}/3$, gives

$$[z^n]F(z) \;\sim\; \sqrt{e}\,\frac{2\sqrt{2}}{3\Gamma(-3/2)}e^n n^{-5/2} \;=\; \sqrt{\frac{e}{2\pi}}e^n n^{-5/2}.$$

Multiplying by $n!$ and applying Stirling's formula gives the number of acyclic graphs on n labelled vertices:

$$a_n \;=\; n![z^n]F(z) \;\sim\; \sqrt{e}\,n^{n-2}.$$

Thus there are asymptotically $\sqrt{e}$ times as many labelled forests as unrooted labelled trees, whose number is exactly n^{n-2}.

The Logarithmic Standard Scale

When analyzing structures that decompose into cycles or when computing expected values, the dominant singularity at $z = 1$ often takes the form of a logarithm, or a product of an algebraic singularity and a logarithm. The Hankel contour techniques can be generalized to evaluate the Cauchy coefficient integrals for these functions.

Theorem 22. Logarithmic Standard Scale [FS09, Theorem VI.2 and pp. 386–388]. For any $\alpha \in \mathbb{R} \setminus \{0, -1, -2, \dots\}$ and any non-negative integer β, the coefficients satisfy

$$[z^n](1-z)^{-\alpha}\left(\log\frac{1}{1-z}\right)^{\beta} \;\sim\; \frac{n^{\alpha-1}}{\Gamma(\alpha)}(\log n)^{\beta}.$$

In the purely logarithmic case where $\alpha = 0$ and β is a positive integer, the leading Hankel contribution vanishes, and we instead have

$$[z^n]\left(\log\frac{1}{1-z}\right)^{\beta} \;\sim\; \frac{\beta}{n}(\log n)^{\beta-1}.$$

Informal Proof: Expand $(\log n - \log w)^\beta$ after $z = 1 - w/n$; Equation (2) controls the tails. For $\alpha \notin \{0, -1, \dots\}$, the leading term gives $(\log n)^\beta/\Gamma(\alpha)$. For $\alpha = 0$, $\int_{\widetilde{\mathcal{H}}_n} (1 - w/n)^{-n-1}\, \mathsf{d}w = \mathcal{O}(R_0^{-n})$, since its primitive $(1 - w/n)^{-n}$ has modulus R_0^{-n} at both endpoints. Exponential decay permits differentiation under Hankel's integral in Theorem 6:

$$\frac{1}{2\pi i} \int_{\mathcal{H}} e^w (-\log w)\, \mathsf{d}w \;=\; \frac{\mathsf{d}}{\mathsf{d}\alpha} \frac{1}{\Gamma(\alpha)} \bigg|_{\alpha=0} \;=\; 1.$$

Here $1/\Gamma(\alpha) = \alpha + \mathcal{O}(\alpha^2)$. Thus the next term $-\beta(\log n)^{\beta-1} \log w$ contributes $\beta n^{-1} (\log n)^{\beta-1}$. □

Theorem 23. Logarithmic Transfer Theorem [FS09, Theorem VI.3]. Let $f(z)$ be Δ-analytic, let $\alpha \in \mathbb{R}$ and let β be a non-negative integer. If, as $z \to 1$ inside the Δ-domain,

$$f(z) \;=\; \mathcal{O}\left((1 - z)^{-\alpha} \left(\log \frac{1}{1 - z} \right)^\beta \right),$$

then

$$[z^n] f(z) \;=\; \mathcal{O}\left(n^{\alpha - 1} (\log n)^\beta \right).$$

Example 24. By Lecture 5, a connected mapping is a directed cycle of rooted labelled trees. Mappings with exactly two components therefore have EGF

$$A(z) \;=\; \frac{1}{2} \left(\log \frac{1}{1 - T(z)} \right)^2, \qquad T(z) \;=\; z e^{T(z)}.$$

Set $x = 1 - ez$ and $L = \log(1/x)$. By Theorem 15, $1 - T(z) = \sqrt{2} x^{1/2} (1 + \mathcal{O}(x^{1/2}))$, hence

$$A(z) \;=\; \frac{1}{2} \left(\frac{L - \log 2}{2} + \mathcal{O}(x^{1/2}) \right)^2 \;=\; \frac{1}{8} (L - \log 2)^2 + \mathcal{O}(x^{1/2} L).$$

Since $1 - T$ is nonzero on the simply connected slit plane $\mathbb{C} \setminus [1/e, \infty)$, its analytic logarithm and hence A continue there (Lecture 6). After rescaling $w = ez$, Theorem 23 with $\alpha = -1/2$ and $\beta = 1$ bounds the remainder's coefficients by $\mathcal{O}(e^n n^{-3/2} \log n)$.

By Theorem 22, $[z^n] L^2 \sim 2e^n \log n / n$; the lower powers of L contribute only $\mathcal{O}(e^n/n)$. Thus, if a_n counts these mappings on n labelled vertices, Stirling's formula gives

$$[z^n] A(z) \;\sim\; \frac{e^n \log n}{4n}, \qquad a_n \;=\; n! [z^n] A(z) \;\sim\; \sqrt{\frac{\pi}{8}}\, n^{n - 1/2} \log n.$$

Bivariate Generating Functions and Expected Values

Bivariate generating functions let us mark a parameter and extract its expected value by differentiation. This often introduces logarithmic factors, making the preceding transfer results useful.

Definition 25. Let $\mathcal{A}$ be a combinatorial class, and let $\chi : \mathcal{A} \to \mathbb{N}$ be a parameter defined on the objects of $\mathcal{A}$. The bivariate ordinary generating function (BOGF) is defined as

$$A(z,u) \;=\; \sum_{a\in\mathcal{A}} z^{|a|}u^{\chi(a)} \;=\; \sum_{n\geq 0,k\geq 0} a_{n,k} z^n u^k,$$

where $a_{n,k}$ is the number of objects in $\mathcal{A}$ of size n having parameter value k. For labelled classes, the bivariate exponential generating function (BEGF) is

$$A(z,u) \;=\; \sum_{n\geq 0,k\geq 0} a_{n,k}\frac{z^n}{n!}u^k.$$

Setting $u = 1$ gives the univariate generating function $A(z, 1)$. If $a_n = \sum_{k\geq 0} a_{n,k}$ is the total number of objects of size n, then $a_n = [z^n]A(z, 1)$ for a BOGF and $a_n = n![z^n]A(z, 1)$ for a BEGF.

For $a_n > 0$, let χ_n be the value of χ on an object chosen uniformly among all objects of size n. Its *expected value* is

$$\mathbb{E}\left[\chi_n\right] \;=\; \frac{1}{a_n}\sum_{k\geq 0} k\cdot a_{n,k}.$$

Differentiating with respect to u and then setting $u = 1$ weights each object by its parameter value. Thus

$$[z^n]\left.\frac{\partial A(z,u)}{\partial u}\right|_{u=1} \;=\; \begin{cases} \displaystyle\sum_{k\geq 0} k\, a_{n,k}, & \text{for a BOGF,} \\[2ex] \displaystyle\frac{1}{n!}\sum_{k\geq 0} k\, a_{n,k}, & \text{for a BEGF.} \end{cases}$$

Dividing by $[z^n]A(z, 1)$ gives the expected value in both cases: for a BEGF, the factor $1/n!$ cancels between numerator and denominator.

Proposition 26. Let $A(z, u)$ be either a BOGF or a BEGF, with u marking a parameter χ. For an object chosen uniformly among all objects of size n, provided $[z^n]A(z, 1) > 0$, the expected value is

$$\mathbb{E}\left[\chi_n\right] \;=\; \frac{[z^n]\left.\frac{\partial A(z,u)}{\partial u}\right|_{u=1}}{[z^n]A(z,1)}.$$

Example 27. Recall from the example on Stirling numbers in Lecture 3 that permutations have BEGF $P(z, u) = (1 - z)^{-u}$, where u marks the number of cycles. Since $[z^n]P(z, 1) = 1$, Proposition 26 and Theorem 22 with $\alpha = \beta = 1$ give the expected number of cycles in a uniformly random permutation:

$$\mathbb{E}\left[\chi_n\right] \;=\; [z^n]\left.\frac{\partial P(z,u)}{\partial u}\right|_{u=1} \;=\; [z^n]\frac{1}{1-z}\log\frac{1}{1-z} \;\sim\; \log n.$$

This recovers the asymptotic growth of the exact expectation H_n found in Lecture 3.

Exercise 28. A simple labelled 2-regular graph is a set of undirected cycles of length at least 3, so $\mathcal{G} = \text{SET}(\mathcal{C})$. The component EGF is

$$C(z) \;=\; \frac{1}{2}\left(\log\frac{1}{1-z} - z - \frac{z^2}{2}\right).$$

Formulate the BEGF $G(z,u)$ where u marks the number of connected components. Find the asymptotic behavior of both $[z^n]G(z,1)$ and $[z^n]\left.\frac{\partial G(z,u)}{\partial u}\right|_{u=1}$ using Theorems 9, 22 and 23. Show that a uniformly chosen such graph on n vertices has asymptotically $\frac{1}{2}\log n$ connected components in expectation.

Example 29. Choose T_n uniformly among the C_n complete ordered binary trees with $n \geq 1$ internal nodes, where C_n is the n-th Catalan number. External leaves mark empty child positions and are not counted. Let $p(t)$ be the sum of the internal nodes' depths, measured in edges from the root at depth 0. We seek the expected average depth $\mathbb{E}\,[p(T_n)/n]$.

If t has subtrees t_L, t_R, then $p(t) = p(t_L) + p(t_R) + |t_L| + |t_R|$: attaching the subtrees increases every internal node's depth by one. With a single external leaf contributing 1, the BOGF satisfies

$$F(z,u) \;=\; \sum_t z^{|t|}u^{p(t)} \;=\; 1 + zF(zu,u)^2.$$

The substitution $z \mapsto zu$ accounts for the additional depth. At $u = 1$, we recover the Catalan OGF $C(z) = F(z,1) = (1-\sqrt{1-4z})/(2z)$.

We need only the derivative $S(z) = \partial F(z,u)/\partial u|_{u=1}$, whose coefficient of z^n sums $p(t)$ over trees with n internal nodes. Differentiating the functional equation gives

$$S(z) \;=\; 2zC(z)\big(zC'(z) + S(z)\big), \qquad S(z) \;=\; \frac{2z^2C(z)C'(z)}{1-2zC(z)}.$$

Writing $s = \sqrt{1-4z}$, we have $C(z) = 2/(1+s)$ and $C'(z) = 4/(s(1+s)^2)$, so

$$S(z) \;=\; \frac{(1-s)^2}{s^2(1+s)} \;=\; \frac{1}{1-4z} + \mathcal{O}\left((1-4z)^{-1/2}\right).$$

This expression is analytic in a scaled Δ-domain at $1/4$. After rescaling $w = 4z$, Theorem 9 gives $[z^n]S(z) = 4^n + \mathcal{O}(4^n n^{-1/2})$. By Proposition 26 and $C_n \sim 4^n/(\sqrt{\pi}\,n^{3/2})$ (Example 10),

$$\mathbb{E}\left[\frac{p(T_n)}{n}\right] \;=\; \frac{[z^n]S(z)}{nC_n} \;\sim\; \sqrt{\pi n}.$$

References

[FO90] Philippe Flajolet and Andrew Odlyzko. Singularity analysis of generating functions. *SIAM Journal on Discrete Mathematics*, 3(2):216–240, 1990.

[FS09] Philippe Flajolet and Robert Sedgewick. *Analytic Combinatorics*. Cambridge University Press, 2009.